ALGEBRAIC CYCLES AND THE TRIANGULATED CATEGORY OF MIXED MOTIVES

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Abstract. In this paper, we shall give a candidate for the t-structure on the triangulated category of mixed motives due to Voevodsky. Through the regulator map, this gives rise to the generalization of the category of mixed Hodge structures which is constructed in [M].

1. Introduction

Let k be a field of characteristic 0 and $\mathcal{L}(k)$ denote the category of the smooth schemes over k. Voevodsky constructs the triangulated category of mixed motives with coefficients in $\mathbb{Q}$ (denoted by $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$) such that we have a canonical functor

$$M : \mathcal{L}(k) \to DM_{\mathrm{gm}}(k)_{\mathbb{Q}}, \quad X \mapsto M(X).$$

It has been believed that a good t-structure on $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ would capture the mixed motives of Grothendieck and this is one of the most significant problems in the field of algebraic cycles.

Remark 1.1. Although other triangulated categories of mixed motives are constructed ([H], [L]), we shall use the category of Voevodsky since it is widely spread and more accessible than others.

2. Review of t-structures

For a triangulated category $\mathbb{D}$, let $\mathbb{D}^{\leq 0}$ and $\mathbb{D}^{\geq 0}$ be full subcategories of $\mathbb{D}$. We say that the couple $(\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 0})$ is a t-structure on $\mathbb{D}$ if the following conditions are satisfied:

(1) $\mathbb{D}^{\leq 0}[1] \subset \mathbb{D}^{\leq 0}$ and $\mathbb{D}^{\geq 0}[-1] \subset \mathbb{D}^{\geq 0}$
(2) $\mathrm{Hom}_{\mathbb{D}}(X, Y) = 0$ for $X \in \mathbb{D}^{\leq 0}$ and $Y \in \mathbb{D}^{\geq 0}[-1]$
(3) For any $X \in \mathbb{D}$, there exists a distinguished triangle $X_0 \to X \to X_1 \to$ in $\mathbb{D}$ such that we have $X_0 \in \mathbb{D}^{\leq 0}$ and $X_1 \in \mathbb{D}^{\geq 0}[-1]$.

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The full subcategory $\mathcal{A} = \mathbb{D}^{\leq 0} \cap \mathbb{D}^{\geq 0}$ is called the heart of the t-structure. In the next section, we shall give a t-structure on $\mathbb{D} = DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ which reflects the intersections of algebraic cycles and we expect that the heart of this t-structure is the mixed motives which Grothendieck has dreamed of.

3. Definition of a t-structure

Keep the notation as in the Introduction and denote $\mathbb{D} = DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ for simplicity. Let us define a t-structure $(\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 0})$ on $\mathbb{D}$ as follows.

Preliminary

Let $\phi : A \to M(X)$ and $\psi : M(Y) \to B$ be two morphisms in $\mathbb{D}$. If we convert the orientations of X and Y, we denote the corresponding morphisms by $\phi^X : A \to M(X)$ and $\psi_Y : M(Y) \to B$.

Definition of $\mathbb{D}^{\leq 0}$

The full subcategory $\mathbb{D}^{\leq 0}$ is consisted of objects A of $\mathbb{D}$ such that we have

$$\phi + \phi^X = 0 : A \to M(X)[i] \qquad (\exists X \in \mathcal{L}(k),\ \exists i \in \mathbb{Z}_{\geq 0}).$$

Definition of $\mathbb{D}^{\geq 0}$

The full subcategory $\mathbb{D}^{\geq 0}$ is consisted of objects B of $\mathbb{D}$ such that we have

$$\psi - \psi_Y = 0 : M(Y)[j] \to B \qquad (\forall Y \in \mathcal{L}(k),\ \forall j \in \mathbb{Z}_{<0}).$$

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