ON THE MATHEMATICAL INTERPRETATIONS OF QUANTUM FIELD THEORY

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Abstract. In this paper, we shall give mathematical interpretations of the particle picture by using the momentum expansion of the quantized wave function of a Schrödinger equation and the theory of special relativity. As a result, we will introduce a new notion of time.

1. SPECIAL RELATIVITY

1.1. **An example.** For the inertial frame of reference S on the ground, consider the inertial frame of reference S' moving at the constant speed V e.g. a railway bogie. Here, assume that the stand AB of this bogie is parallel to the direction of motion of S' and that a light equipment is put on the midpoint of AB. Let us turn on the light. Then, it is no wonder that a person on the bogie S' will see that the light arrives at the points A and B at the same time. On the other hand, a person on the ground S will see that, by the principle of invariant light speed, the light arrives at the point A faster than at the point B.

Although turning on the light is the only one event, this example shows that the utterly different worlds exist associated to the different inertial frames of reference. After this, we shall consider the relationships of these different worlds.

1.2. **Principle of invariant light speed.** In the example above, the reason why one see that the different worlds exist associated to the different inertial frames of reference results from the principle of invariant light speed.

$$A \bullet \text{ -- -- }\overset{\text{light}}{\text{-- -- --}}\text{ -- } \succ \qquad \bullet\overset{B}{\Longrightarrow}\text{ -- }\overset{\text{light}}{\text{-- -- --}}\text{ } \succ$$

As the figure above shows, a person on the ground will see that both of the light from the stationary point A and the light from the point B moving at the constant speed V moves at the speed of light (denoted by c) and this contradicts the formula of vectors. Although it is a one way that we take it for granted that it is the nature of light, we will introduce the virtual but mathematical spaces where the formula of vectors can be concluded (see the figure below).
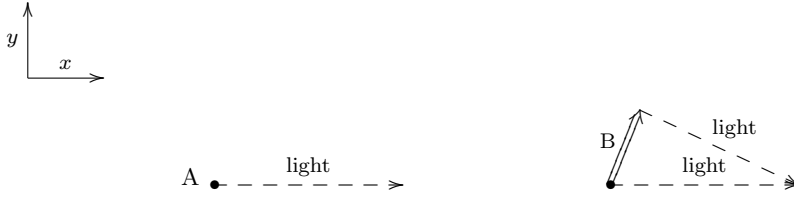
Here, a person on the ground can see only the x-axis and the formula of vectors is concluded in this mathematically virtual world (x-y plane) which is invisible to the person. Even if the point B moves at the more speed, the person on the ground will see that the light from the point B always moves at the speed c. Thus, it is no wonder that, if the point B moves faster, the more momentum is annihilated to the direction of y-axis and this quantity does not contribute to the observation at the stationary point A.

2. Quantum mechanics

2.1. **Wave functions and Stochastic interpretations.** It is believed that the movements of fine particles such as electrons are captured only by the stochastic methods and their stochastic futures are described by the wave functions of Schrödinger equations. In this section, by using the momentum expansions of these (second) quantized wave functions and the discussions given by the theory of special relativity in the previous section, we shall give mathematical interpretations of the particle picture.

2.2. **Quantization and Momentum expansion.** Let us consider a free electron (in the 1-dimensional space) which is not bound to external forces. Let m denote the mass of this electron, x denote the space axis and t denote the time axis. Then, the wave function $\Psi(x,t)$ of this free electron satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} \quad (\hbar = \frac{h}{2\pi}, \ h : \text{the Planck constant}).$$

By quantizing this wave function $\Psi(x,t)$, we can obtain the quantum field operators $\phi(x,t)$. By this operator $\phi(x,t)$, a particle at (x,t) is annihilated and on the other hand, by its Hermite operator $\phi^\dagger(x,t)$, a particle at (x,t) is created. The Fourier expansion of this $\phi(x,t)$ is given by

$$(*) \qquad \phi(x,t) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \exp[i(-\omega_\mathbf{p} t + \mathbf{p}x)]a_\mathbf{p}.$$

Here, $\hbar\omega_\mathbf{p} = \frac{\hbar^2\mathbf{p}^2}{2m}$ denotes the energy of the free electron with the momentum $\hbar\mathbf{p}$. Then, the operators $a_\mathbf{p}$ and $a_\mathbf{q}^\dagger$ satisfy the anti-commutative relation $\{a_\mathbf{p}, a_\mathbf{q}^\dagger\} = \delta(\mathbf{p} - \mathbf{q})$ and one can see that $a_\mathbf{p}$ and $a_\mathbf{p}^\dagger$ are annihilation-creation operators of a particle with the momentum $\hbar\mathbf{p}$. Thus, the Fourier expansion above is the momentum expansion by these operators.

As mentioned above, ϕ and $\phi^\dagger$ preside the annihilation-creation of one particle. On the other hand, as is discussed in the Section 1.1, each different world exists associated to the inertial frame of reference moving at each speed (momentum) around this free electron. Putting these two things together, we can interpret that the momentum expansion ($*$) is the sum of the annihilation-creation operators in each world and as a whole, this is the annihilation-creation operators of one particle. Furthermore, in the Section 1.2, let the point A denote the stationary free electron and the point B denote a moving inertial frame of reference. Then, if B increases the momentum, a person at the point A see that the increased momentum is annihilated. From the momentum expansion ($*$), we can interpret that such increased momentum contributes to the annihilation operator $\phi(x, t)$.

3. Some comments

3.1. **Uncertainty principle.** It is assumed that we cannot determine both of the position and the momentum of a fine particle at once. By the mathematical interpretations in the Section 2.2, however, we arrive at the new idea that each different world exists associated to the inertial frame of reference moving each momentum around a fine particle and that these worlds with certainty (where we can determine both of the position and the momentum of a fine particle at once) contribute to the description of a fine particle as a wave.

3.2. **Deformation theory of wave functions.** The q-expansion of a new modular form of weight 2 contains the information about the number of $\mathbb{F}_p$-rational points on an elliptic curve over $\mathbb{Z}$ modulo p. For each prime number p, the coefficient a_p of this q-expansion reflects this information and I studied the deformation of these $\{a_p\}$ ([Mo2]). We will give some observations on the deformation of wave functions by considering the similarities between modular forms and wave functions.

As mentioned in the previous section, we can interpret that each different world exists associated to the inertial frame of reference moving with each momentum $\hbar\mathbf{p}$. This is similar to the fact that, in number theory, there exists each different world for each prime number p and the q-expansion of a new modular form is the sum of the coefficients $\{a_p\}$ of these worlds. Let us consider the deformation of the coefficients $\{a_\mathbf{p}\}$ of the momentum expansion.

Let f and g denote the wave functions of two same kind fine particles and assume that one can deform f to g continuously without changing the energy. Since each different world exists associated to the inertial frame of reference moving with each momentum, we can suggest that two particles whose states are represented by f and g are observed as two futures (in two different worlds) of one particle. This may be a mathematical explanation of the indistinguishability of particles which is one of the principles of quantum mechanics.

References

[Mo1]   Morita, K.:*Generalization of the theory of mixed Hodge structures and its application.*
[Mo2]   Morita, K.:*On the topological aspects of arithmetic elliptic curves.*

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