GALOIS COHOMOLOGY OF A p-ADIC FIELD VIA (Φ,Γ)-MODULES IN THE IMPERFECT RESIDUE FIELD CASE

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ABSTRACT. For a p-adic local field K with perfect residue field, L. Herr constructed a complex which computes the Galois cohomology of a p-torsion representation of the absolute Galois group of K by using the theory of (Φ,Γ)-modules. We shall generalize his work to the imperfect residue field (the residue field has a finite p-basis) case.

1. INTRODUCTION

In this article, K denotes a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p>0$ such that $[k:k^p]=p^n<\infty$. Assume that K contains a primitive p-th root of unity if $p\neq 2$ and a primitive 4-th root of unity if $p=2$. Choose an algebraic closure $\overline{K}$ of K and put $G_K=\mathrm{Gal}(\overline{K}/K)$. By a p-torsion G_K-representation, we mean a $\mathbb{Z}_p$-module of finite length endowed with a continuous action of G_K. Let $\mathbf{Rep}_{p-\mathrm{tor}}(G_K)$ denote the category of p-torsion G_K-representations. Let V be a p-torsion G_K-representation. In the case $n=0$ (i.e. k is a perfect field), Herr [H1] obtained a presentation of the Galois cohomology $H^*(G_K,V)$ in terms of the (Φ,Γ_K)-module $D(V)$ associated to V in the sense of Fontaine [F].

Now, let n be arbitrary. The purpose of this paper is to give a presentation of $H^*(G_K,V)$ in terms of the (Φ,Γ_K)-module (defined in this paper) associated to V (Theorem 1.1). Our Γ_K is non-commutative if $n\geq 1$.

Fix a lifting $(b_i)_{1\leq i\leq n}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K), and for each $m\geq 1$ and $1\leq i\leq n$, fix a p^m-th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p=b_i^{1/p^m}$. Put $K^{(\prime)}=\cup_{m\geq 0}K(b_i^{1/p^m},1\leq i\leq n)$ and $K_\infty^{(\prime)}=\cup_{m\geq 0}K^{(\prime)}(\zeta_{p^m})$ where ζ_{p^m} denotes a primitive p^m-th of unity in $\overline{K}$ such that $\zeta_{p^{m+1}}^p=\zeta_{p^m}$. The field $K^{(\prime)}$ depends on the choice of a lifting of a p-basis of k in $\mathcal{O}_K$, but the field $K_\infty^{(\prime)}$ doesn't. Let K' denote the p-adic completion of $K^{(\prime)}$. Choose an algebraic closure $\overline{K'}$ ($\supset\overline{K}$) of K'. Put $K'_\infty=\cup_{m\geq 0}K'(\zeta_{p^m})$ in $\overline{K'}$. These fields K' and K'_∞ depend on the choice of a lifting of a p-basis of k in $\mathcal{O}_K$. Put $\Gamma_K=\mathrm{Gal}(K_\infty^{(\prime)}/K)$ and $\Gamma_{K'}=\mathrm{Gal}(K'_\infty/K')$. Then, $\Gamma_{K'}$ is isomorphic to an open subgroup of $\mathbb{Z}_p^*$ via

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the cyclotomic character $\chi : \Gamma_{K'} \to \mathbb{Z}_p^*$ and Γ_K is isomorphic to the semi-direct product $\Gamma_{K'} \ltimes \mathbb{Z}_p^{\oplus n}$ where $\Gamma_{K'}$ acts on $\mathbb{Z}_p^{\oplus n}$ via χ (see Section 3). The group Γ_K is non-commutative if $n \geq 1$. Since K' has perfect residue field which we denote $k' = k^{p^{-\infty}}$, we can apply the theory of Fontaine [F] to obtain the $(\Phi, \Gamma_{K'})$-module $D(V)$ for a p-torsion G_K-representation V. Then, $D(V)$ is equipped with a Frobenius operator $\phi : D(V) \to D(V)$ and also with a continuous action of Γ_K (not only $\Gamma_{K'}$) which commutes with ϕ. With these actions , $D(V)$ becomes an object of the category $\Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$ of torsion étale (Φ, Γ_K)-modules which we will define (see Section 2) by imitating the definition of the category of torsion étale $(\Phi, \Gamma_{K'})$-modules by Fontaine ([F], p273, 3.3.2). Then, we shall obtain an equivalence of categories between

$$\mathbf{Rep}_{p\text{-tor}}(G_K) \quad \text{and} \quad \Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$$

which is a generalization of the equivalence of Fontaine ([F], p274, 3.4.3) to the imperfect residue field case (for details, see Theorem 2.7 in Section 2). By using this $D(V)$, we will construct a complex $C_{\phi,\Gamma_K}(D(V))$ in Section 3. Our main result is the following.

Theorem 1.1. *With notations as above, the group $H^i(G_K, V)$ is canonically isomorphic to the i-th cohomology group of the complex $C_{\phi,\Gamma_K}(D(V))$ for all i. This isomorphism is functorial in V.*

Our proof of the main theorem is a little different from the method of Herr. In the case $n = 0$, he considered an "effaceable" property of the complex $C_{\phi,\Gamma_K}(D(V))$, whereas our method is to construct a free resolution of the $\mathbb{Z}_p[[\Gamma_K]]$-module $\mathbb{Z}_p$.

This paper is organized as follows. In Section 2, we shall review the theory of (Φ, Γ)-modules, which is due to J.-M. Fontaine [F] in the perfect residue field case. We shall construct a theory of (Φ, Γ)-modules in the imperfect residue field case (F. Andreatta [A] constructs a more general and finer theory of (Φ, Γ)-modules). In Section 3, for $M \in \Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$, we shall construct the complexes $C_{\Gamma_K}(M)$ and $C_{\phi,\Gamma_K}(M)$ which are to be used in the main theorem. In Section 4, we shall construct a free resolution of $\mathbb{Z}_p$ in the category of left $\mathbb{Z}_p[[\Gamma_K]]$-modules. In Section 5, we shall prove that the cohomology group of $C_{\phi,\Gamma_K}(D(V))$ coincides with the Galois cohomology $H^*(G_K, V)$.

2. The theory of (Φ, Γ)-modules

Let k' denote the perfect residue field of K' as in Section 1. Put $F(k') = W(k')[p^{-1}]$ where $W(k')$ denotes the ring of Witt vectors with coefficients in k'. Now, we apply the theory of (Φ, Γ)-modules of Fontaine to K'. Since $K^{(\prime)}$ is a Henselian discrete valuation field , we have an isomorphism $G_{K'} = \mathrm{Gal}(\overline{K'}/K') \simeq G_{K^{(\prime)}} = \mathrm{Gal}(\overline{K}/K^{(\prime)})$ $(\subset G_K)$. With this isomorphism, we identify $G_{K'}$ with a subgroup of G_K. We have a bijective map from the set of finite extensions of $K^{(\prime)}$ contained in $\overline{K}$ to the set of finite extensions of K' contained in $\overline{K'}$ defined by

$L \to LK'$. Furthermore, LK' is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K'}}/p^n\mathcal{O}_{\overline{K'}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K'}}$ denote the rings of integers of $\overline{K}$ and $\overline{K'}$. Thus, the p-adic completion of $\overline{K}$ is isomorphic to the p-adic completion of $\overline{K'}$, which we will write $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \cdots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}$$

and $\widetilde{\mathbb{E}}^+$ denotes the set of $x = (x^{(i)}) \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$ (where $\mathcal{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$). For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, define their sum and product by $(x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. Let $\epsilon = (\epsilon^{(i)})$ denote an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is a field of characteristic $p > 0$ ($\widetilde{\mathbb{E}}^+$ is a subring of $\widetilde{\mathbb{E}}$) and is the completion of an algebraic closure of $k'((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$.

Example 2.1. With this valuation, we have

$$v_{\mathbb{E}}(\epsilon - 1) = \lim_{n\to\infty} v_p((\epsilon^{(n)} - 1)^{p^n}) = \frac{p}{p-1}.$$

The field $\widetilde{\mathbb{E}}$ is equipped with an action of a Frobenius σ and a continuous action of the Galois group G_K with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Put $\mathbb{E}_{F(k')} = k'((\epsilon - 1))$ and define $\mathbb{E}$ to be the separable closure of $\mathbb{E}_{F(k')}$ in $\widetilde{\mathbb{E}}$. Define $H_K = \mathrm{Gal}(\overline{K'}/K'_\infty)$ which is isomorphic to the subgroup $G_{K_\infty^{(\prime)}} = \mathrm{Gal}(\overline{K}/K_\infty^{(\prime)})$ of G_K. From now on, we identify H_K with $G_{K_\infty^{(\prime)}}$. If we put $\mathbb{E}_K = \mathbb{E}^{H_K}$ and define $G_{\mathbb{E}_K}$ to be the Galois group of $\mathbb{E}/\mathbb{E}_K$, the action of $G_{K'}$ on $\mathbb{E}$ induces the canonical isomorphism $H_K \simeq G_{\mathbb{E}_K}$ by the theory of the field of norms ([FW], [W]). Let π denote $[\epsilon] - 1$. Put $\widetilde{\mathbb{A}} = W(\widetilde{\mathbb{E}})$ ($\widetilde{\mathbb{E}}$ is a perfect field) and $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$. The ring $\widetilde{\mathbb{A}}$ is endowed with the topology whose fundamental system of the neighborhoods of 0 has the form $\pi^k\widetilde{\mathbb{A}}^+ + p^{n+1}\widetilde{\mathbb{A}}$ for $k, n \in \mathbb{N}$. This topology coincides with the product topology defined by the application $\widetilde{\mathbb{A}} \to \widetilde{\mathbb{E}}^{\mathbb{N}} : x \mapsto (x_k)_{k\in\mathbb{N}}$. The continuous action of G_K on $\widetilde{\mathbb{E}}$ induces the continuous action of G_K on $\widetilde{\mathbb{A}}$ which commutes with the Frobenius σ. Let $\mathbb{A}_{F(k')}$ be the p-adic completion of $W(k')[[\pi]][\pi^{-1}]$ contained in $\widetilde{\mathbb{A}}$. This ring is a complete discrete valuation ring with the residue field $\mathbb{E}_{F(k')}$. Let $\mathbb{A}$ be the p-adic completion of the maximal unramified extension of $\mathbb{A}_{F(k')}$ in $\widetilde{\mathbb{A}}$ which has the residue field $\mathbb{E}$. The ring $\mathbb{A}$ is equipped with an action of the Galois group G_K and of the Frobenius σ induced from those of $\widetilde{\mathbb{E}}$. Put $\mathbb{A}_K = \mathbb{A}^{H_K}$.

For all $V \in \mathbf{Rep}_{p-\mathrm{tor}}(G_{K'})$, we can associate the $(\Phi, \Gamma_{K'})$-module over $\mathbb{A}_K$

$$D(V) = (\mathbb{A} \otimes_{\mathbb{Z}_p} V)^{H_K}.$$

It is equipped with the residual action of $\Gamma_{K'} \simeq G_{K'}/H_K$ and the Frobenius $\phi_{D(V)}$ induced by that on $\mathbb{A}$. The module $D(V)$ is a torsion étale $(\Phi, \Gamma_{K'})$-module over $\mathbb{A}_K$ ([F], p274, 3.4.2).

Conversely, to a torsion étale $(\Phi, \Gamma_{K'})$-module M over $\mathbb{A}_K$, we can associate a p-torsion representation of $G_{K'}$ as follows

$$(*) \quad V(M) = (\mathbb{A} \otimes_{\mathbb{A}_K} M)^{\sigma \otimes \phi_M = 1} \in \mathbf{Rep}_{p-\mathrm{tor}}(G_{K'}).$$

Let $\Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_{K'}}$ denote the category of torsion étale $(\Phi, \Gamma_{K'})$-modules in the sense of Fontaine ([F], p273, 3.3.2). By the two constructions above, Fontaine proved the following ([F], p274, 3.4.3).

Theorem 2.2. *The functor D gives an equivalence between the two categories*

$$\mathbf{Rep}_{p-tor}(G_{K'}) \quad \text{and} \quad \Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_{K'}}$$

The functor V is a quasi-inverse of D.

Define (Φ, Γ_K)-modules as follows.

Definition 2.3. A torsion (Φ, Γ_K)-module over $\mathbb{A}_K$ is an $\mathbb{A}_K$-module M of finite length equipped with

(1) a σ-semi-linear map (which we call a Frobenius operator)

$$\phi = \phi_M : M \to M$$

(2) a continuous semi-linear action of Γ_K which commutes with ϕ.

In addition, we call M an étale (Φ, Γ_K)-module if it is generated by the image of ϕ as an $\mathbb{A}_K$-module. Let $\Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$ denote the category which consists of

∘ objects: torsion étale (Φ, Γ_K)-modules over $\mathbb{A}_K$

∘ morphisms: $\mathbb{A}_K$-linear morphisms which commute with ϕ and the action of Γ_K.

Remark 2.4. Put $\mathbb{A}_K^+ = \mathbb{A}_K \cap \widetilde{\mathbb{A}}^+$. If we fix a lifting T_K of the prime element of $\mathbb{E}_K$ in $\mathbb{A}_K^+$, we have $\mathbb{A}_K^+ = W(k')[[T_K]]$. Let M be a finitely generated $\mathbb{A}_K/p^n$-module. Fix a finitely generated sub-$\mathbb{A}_K^+/p^n$-module M^0 of M such that M is generated by M^0 over $\mathbb{A}_K/p^n$. The module M is endowed with the topology such that the family of submodules $\{T_K^m M^0\}_{m\geq 1}$ is a fundamental system of neighborhoods of 0. This topology is independent of the choice of M^0. Furthermore, since $\mathbb{A}_K/p^n$ is Noetherian and complete for the T_K-adic topology, $T_K^{-N} M^0$ is complete for the T_K-adic topology. We may use the family of submodules $\{\pi^m M^0\}_{m\geq 1}$ instead of $\{T_K^m M^0\}_{m\geq 1}$ to define the same topology.

Consider a p-torsion G_K-representation $V \in \mathbf{Rep}_{p-\mathrm{tor}}(G_K)$ and $D(V)$. Since the Galois group G_K acts on $\mathbb{A} \otimes_{\mathbb{Z}_p} V$ and we have $D(V) = (\mathbb{A} \otimes_{\mathbb{Z}_p} V)^{H_K}$, the quotient $\Gamma_K \simeq G_K/H_K$ and ϕ act on $D(V)$ commuting with each other. This means that $D(V)$ becomes an object of $\Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$. The continuous action of G_K on $\mathbb{A}\otimes_{\mathbb{Z}_p}V$ induces the continuous action of Γ_K on $D(V)$ as follows. Let L be a finite Galois extension of K contained in $\overline{K}$ such that the action of $G_L=\mathrm{Gal}(\overline{K}/L)$ on V is trivial. Fix $n\in\mathbb{N}$ such that $p^nV=0$. Then, we have $D(V)=(\mathbb{A}_L/p^n\otimes_{\mathbb{Z}_p}V)^{H_K}$. The following two topologies of $D(V)$ coincide

(1) the topology defined in Remark 2.4
(2) the induced topology as a subspace of $\mathbb{A}_L/p^n\otimes_{\mathbb{Z}_p}V$ whose topology is defined in Remark 2.4.

(Proof: There exists an $\mathbb{A}_L/p^n$-linear isomorphism

$$\mathbb{A}_L/p^n\otimes_{\mathbb{A}_K/p^n}D(V)\simeq\mathbb{A}_L/p^n\otimes_{\mathbb{Z}_p}V.$$

Fix a finitely generated sub-$\mathbb{A}_K^+/p^n$-module M^0 of $D(V)$ such that $D(V)$ is generated by M^0 over $\mathbb{A}_K/p^n$. Let M_L^0 be the sub-$\mathbb{A}_L^+/p^n$-module of $\mathbb{A}_L/p^n\otimes_{\mathbb{Z}_p}V$ generated by M^0. Since the morphism $\mathbb{A}_K^+/p^n\to\mathbb{A}_L^+/p^n$ is finite flat, the morphism $\mathbb{A}_L^+/p^n\otimes_{\mathbb{A}_K^+/p^n}M^0\to M_L^0$ is an isomorphism. Thus, the inverse image of $\pi^mM_L^0$ by the map $D(V)\to\mathbb{A}_L/p^n\otimes_{\mathbb{Z}_p}V$ is π^mM^0.)

Remark 2.5. Let L be a finite Galois extension of K contained in $\overline{K}$. Let M be a finitely generated $\mathbb{A}_L/p^n$-module endowed with a continuous and semi-linear action of $\mathrm{Gal}(K_\infty^{(\prime)}L/K)$. Fix M^0 as in Remark 2.4. Let M^1 be the sub-$\mathbb{A}_L^+/p^n$-module of M generated by $g(M^0)$ ($g\in\mathrm{Gal}(K_\infty^{(\prime)}L/K)$). Since $\mathrm{Gal}(K_\infty^{(\prime)}L/K)$ is compact, M^1 is also a finitely generated sub-$\mathbb{A}_L^+/p^n$-module. By construction, M^1 is stable under the action of $\mathrm{Gal}(K_\infty^{(\prime)}L/K)$. Then, for $N,m\in\mathbb{N}$, $\pi^{-N}M^1/\pi^mM^1$ becomes a discrete $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{(\prime)}L/K)]]$-module. Since $\pi^{-N}M^1$ is complete for the π-adic topology, $\pi^{-N}M^1$ has the structure of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{(\prime)}L/K)]]$- module. Thus, M is equipped with the structure of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{(\prime)}L/K)]]$-module. For another sub-$\mathbb{A}_L^+/p^n$-module M^2 of M stable under the action of $\mathrm{Gal}(K_\infty^{(\prime)}L/K)$ such that M is generated by M^2 over $\mathbb{A}_L/p^n$, we can find integers $N_1\leq N_2$ such that $\pi^{N_1}M^1\subset M^2\subset\pi^{N_2}M^1$, therefore, the structure of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{(\prime)}L/K)]]$-module on M is independent of the choice of M^1. With this, the action of Γ_K on $D(V)$ naturally extends to the action of $\mathbb{Z}_p[[\Gamma_K]]$.

Remark 2.6. Let L be a finite Galois extension of K contained in $\overline{K}$ such that the action of G_L on V is trivial. Fix $n\in\mathbb{N}$ such that $p^nV=0$. For a finite Galois extension of K such that $L\subset L'\subset\overline{K}$, $\mathbb{A}_{L'}\otimes_{\mathbb{Z}_p}V$ is a finitely generated $\mathbb{A}_{L'}/p^n$-module endowed with a continuous and semi-linear action of $\mathrm{Gal}(K_\infty^{(\prime)}L'/K)$. Remark 2.5 says that the action of $\mathrm{Gal}(K_\infty^{(\prime)}L'/K)$ on $\mathbb{A}_{L'}\otimes_{\mathbb{Z}_p}V$ naturally extends to the action of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{(\prime)}L'/K)]]$. Thus, the action of G_K on $\mathbb{A}\otimes_{\mathbb{Z}_p}V=\varinjlim_{L'}(\mathbb{A}_{L'}\otimes_{\mathbb{Z}_p}V)$ naturally extends to the action of $\mathbb{Z}_p[[G_K]]$. Then, the canonical injection $D(V)\to\mathbb{A}\otimes_{\mathbb{Z}_p}V$ is compatible with the action of $\mathbb{Z}_p[[G_K]]$. (Proof: Let L, M^0, M_L^0 be as in the proof of the coincidence of the two topologies of $D(V)$ before Remark 2.5. We can assume that M^0 is endowed with a continuous and semi-linear action of Γ_K (see Remark 2.5). Since the morphism $\mathbb{A}_K^+/p^n \to \mathbb{A}_L^+/p^n$ is finite flat, we have a morphism of discrete $\mathbb{Z}_p[[G_K]]$-modules $\pi^{-N}M^0/\pi^m M^0 \to (\mathbb{A}_L^+/p^n)\otimes_{\mathbb{A}_K^+/p^n}(\pi^{-N}M^0/\pi^m M^0) \simeq \pi^{-N}M_L^0/\pi^m M_L^0$. By taking the inverse limit for m, we obtain the morphism of $\mathbb{Z}_p[[G_K]]$-modules $\pi^{-N}M^0 \to \pi^{-N}M_L^0$.)

Conversely, to a torsion étale (Φ, Γ_K)-module M over $\mathbb{A}_K$, we can associate a p-torsion representation of G_K as follows (see $(*)$)

$$V(M) = (\mathbb{A}\otimes_{\mathbb{A}_K} M)^{\sigma\otimes\phi_M=1} \in .$$

The continuous action of G_K on $\mathbb{A}\otimes_{\mathbb{A}_K} M$ induces the continuous action of G_K on $V(M)$. Here, we give $V(M)$ the induced topology as a subspace of $\mathbb{A}\otimes_{\mathbb{A}_K} M$. Since the topology of $\mathbb{A}\otimes_{\mathbb{A}_K} M$ is Hausdorff and $V(M)$ is finite, the induced topology on $V(M)$ is discrete.

An imperfect residue field version of Fontaine's theorem is the following (cf. Theorem 2.2).

Theorem 2.7. *The functor D gives an equivalence between the two categories*

$$\mathbf{Rep}_{p\text{-}tor}(G_K) \quad \text{and} \quad \Phi\Gamma\mathbf{M}^{\acute{e}t,p\text{-}tor}_{\mathbb{A}_K,\Gamma_K}$$

The functor V is a quasi-inverse of D.

Proof. For $M \in \Phi\Gamma\mathbf{M}^{\text{ét},p\text{-tor}}_{\mathbb{A}_K,\Gamma_K}$, the natural morphism

$$\mathbb{A}\otimes_{\mathbb{Z}_p} V(M) \to \mathbb{A}\otimes_{\mathbb{A}_K} M$$

induces a morphism $D(V(M)) \to M$ and this morphism is an isomorphism ([F], p258, 1.2.6). Conversely, for $N \in \mathbf{Rep}_{p\text{-tor}}(G_K)$, the natural morphism

$$\mathbb{A}\otimes_{\mathbb{A}_K} D(N) \to \mathbb{A}\otimes_{\mathbb{Z}_p} N$$

induces a morphism $V(D(N)) \to N$ and this morphism is an isomorphism ([F], p258, 1.2.4). □

3. Main theorem

We will give a presentation of $H^*(G_K, V)$ in terms of $D(V)$. Recall that we fixed a p^m-th root b_i^{1/p^m} of b_i in Introduction. Fix a p^m-th root ζ_{p^m} of unity such that $\zeta_{p^{m+1}}^p = \zeta_{p^m}$. Fix a topological generator γ of $\Gamma_{K'} \subset \Gamma_K$ and define $\beta_i \in \Gamma_K$ $(1 \le i \le n)$ by

$$\beta_i(b_i^{1/p^m}) = b_i^{1/p^m}\zeta_{p^m},\ \beta_i(b_j^{1/p^m}) = b_j^{1/p^m}\ (j \neq i) \text{ and } \beta_i(\zeta_{p^m}) = \zeta_{p^m}.$$

Define $l \in \mathbb{Z}_p^*$ by

$$\gamma(\zeta_{p^m}) = \zeta_{p^m}^l.$$

These topological generators $(\gamma, \beta_1, \dots, \beta_n)$ define the isomorphism $\Gamma_K \simeq \Gamma_{K'} \ltimes \mathbb{Z}_p^{\oplus n}$ ($\beta_i \mapsto$ the topological generator of i-th component of $\mathbb{Z}_p$). Let Λ denote $\mathbb{Z}_p[[\Gamma_K]]$ in what follows. Define elements of Λ as follows

$$\omega_i = \beta_i - 1 \quad \text{and} \quad \tau_S = (\Pi_{i\in S}\frac{\beta_i - 1}{\beta_i^l - 1})\gamma - 1.$$

Recall that $D(V)$ is naturally equipped with the action of Λ (Remark 2.5). Since $(\beta_i^l - 1)(\beta_i - 1)^{-1} = \{(1+\omega_i)^l - 1\}\omega_i^{-1} \in l + \omega_i\mathbb{Z}_p[[\omega_i]]$ and $l \in \mathbb{Z}_p^*$, we have $(\beta_i^l - 1)(\beta_i - 1)^{-1} \in \mathbb{Z}_p[[\omega_i]]^*$.

(1) **The complex $C_{\Gamma_K}(D(V))$**

To a p-torsion representation V of G_K, define the complex $C_{\Gamma_K}(D(V))$ to be

$$0 \longrightarrow D(V)^{X(0)} \xrightarrow{d^0} D(V)^{X(1)} \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} D(V)^{X(i)} \xrightarrow{d^i} \cdots \xrightarrow{d^n} D(V)^{X(n+1)} \longrightarrow 0.$$

(The proof of $d^i \circ d^{i-1} = 0$ follows from the presentation of $C_{\Gamma_K}(D(V))$ in terms of C_Λ in Section 5.)

Here

(a) For a finite set X, we define $D(V)^X = \oplus_{S\in X} D(V)$.

(b) $X(i)$ denotes the set of all subsets of $\{0, \cdots, n\}$ of order i. Notice that the order of $X(i)$ is $\binom{n+1}{i}$. We define the degree of $D(V)^{X(0)}$ to be 0.

(c) For $S \in X(i)$ and $T \in X(i+1)$, the (S,T)-component $d^i(S,T)$ of $d^i : D(V)^{X(i)} \to D(V)^{X(i+1)}$ is defined as follows.

(A) If $S \not\subset T$, $d^i(S,T) = 0$.

(B) If $S \subset T$, put $\{j\} = T \backslash S$.

∘ If $j = 0$, $d^i(S,T) = \tau_S$.

∘ If $j \neq 0$, $d^i(S,T) = (-1)^{a(S,j)}\omega_j$ where $a(S,j) = \sharp\{x \in S; x \le j\}$.

(2) **The complex $C_{\phi,\Gamma_K}(D(V))$**

Define the complex $C_{\phi,\Gamma_K}(D(V))$ by

$$C_{\phi,\Gamma_K}(D(V)) = \text{ the mapping fiber of } C_{\Gamma_K}(D(V)) \xrightarrow{\rho} C_{\Gamma_K}(D(V))$$

where $\rho = \phi - 1$. The complex $C_{\phi,\Gamma_K}(D(V))$ has the following form

$$0 \longrightarrow D(V)^{\oplus\binom{n+2}{0}} \xrightarrow{d^0} D(V)^{\oplus\binom{n+2}{1}} \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} D(V)^{\oplus\binom{n+2}{i}} \xrightarrow{d^i} \cdots \xrightarrow{d^{n+1}} D(V)^{\oplus\binom{n+2}{n+2}} \longrightarrow 0.$$

(Here, define the degree of $D(V)^{\oplus\binom{n+2}{0}}$ to be 0.)

Our main result is the following.

Theorem 3.1. *With notations as above, the group $H^i(G_K, V)$ is canonically isomorphic to the i-th cohomological group of the complex $C_{\phi,\Gamma_K}(D(V))$ for all i. This isomorphism is functorial in V.*

It follows that the cohomological dimension of K is $n+2$.

Example 3.2. (1) The case $n = 0$ (i.e. the residue field k is perfect)
In this case, the complex $C_{\phi,\Gamma_K}(D(V))$ is given by

$$0 \longrightarrow D(V) \xrightarrow{d^0} D(V) \oplus D(V) \xrightarrow{d^1} D(V) \longrightarrow 0$$

$\circ$ $d^0(x) = (\rho(x), \tau(x))$,

$\circ$ $d^1(x, y) = (\rho(y) - \tau(x))$.

Here, $\rho = \phi - 1$ and $\tau = \tau_\varnothing = \gamma - 1$. This is the complex constructed by Herr.

(2) The case $n = 1$ (i.e. the residue field k is imperfect and $[k : k^p] = p$)
In contrast to the example (1), there is an action of $\omega_1 = \beta_1 - 1$. Therefore, we have a more complicated complex than before.

$$0 \longrightarrow D(V) \xrightarrow{d^0} D(V) \oplus D(V) \oplus D(V)$$
$$\xrightarrow{d^1} D(V) \oplus D(V) \oplus D(V) \xrightarrow{d^2} D(V) \longrightarrow 0$$

$\circ$ $d^0(x) = (\rho(x), \tau(x), \omega_1(x))$,

$\circ$ $d^1(x, y, z) = (\rho(y) - \tau(x), \rho(z) - \omega_1(x), \tau_{\{1\}}(z) - \omega_1(y))$,

$\circ$ $d^2(x, y, z) = (\rho(z) - \tau_{\{1\}}(y) + \omega_1(x))$.

The appearance of $\tau_{\{1\}}$, instead of τ, reflects the non-commutativity of Γ_K.

4. Construction of a free resolution of $\mathbb{Z}_p$

4.1. **Relations in Λ.** Let

$$\gamma, \beta_1, \beta_2, \cdots, \beta_n$$

be the topological generators of Γ_K as in the previous section. We have the following relations

(1) $\gamma\beta_i = \beta_i{}^l\gamma$ $(l \in \mathbb{Z}_p^*)$
(2) $\beta_i\beta_j = \beta_j\beta_i$.

For the construction of the complex, consider the following elements of Λ (these are introduced in the previous section)

(1) $\tau = \gamma - 1$
(2) $\omega_i = \beta_i - 1$

(3) $W_i = {\beta_i}^l - 1$
(4) $\tau_S = \prod_{i \in S}(\omega_i W_i^{-1})\gamma - 1$ for $S \subset \{1, \cdots, n\}$.
Recall that $\omega_i W_i^{-1} \in \mathbb{Z}_p[[\beta_i - 1]] \subset \Lambda$.

Remark 4.1. Notice that $\tau_S = \tau$ if $S = \varnothing$.

These operators have the following relations:

Relations (R)

(1) $\omega_i \omega_j = \omega_j \omega_i$
(2) $W_i W_j = W_j W_i$
(3) $\gamma \omega_i = W_i \gamma$
(4) For $i \in S \subset \{1, \cdots, n\}$, $\tau_S \omega_i = \omega_i \tau_{S \backslash \{i\}}$.

Proof. $\tau_S \omega_i$
$= (\prod_{j \in S}(\omega_j W_j^{-1})\gamma - 1)\omega_i = (\prod_{j \in S}(\omega_j W_j^{-1})\gamma\omega_i - \omega_i)$
$= (\prod_{j \in S}(\omega_j W_j^{-1})W_i\gamma - \omega_i) = \omega_i(\prod_{j \in S, j \neq i}(\omega_j W_j^{-1})\gamma - 1)$
$= \omega_i \tau_{S \backslash \{i\}}$. □

4.2. **Construction of C_Λ.** Consider the following sequence C_Λ of left Λ-modules

$$0 \longrightarrow \Lambda^{X(n+1)} \xrightarrow{d_n} \Lambda^{X(n)} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_i} \Lambda^{X(i)} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_0} \Lambda^{X(0)} \longrightarrow 0.$$

Here

(1) For a finite set X, define $\Lambda^X = \oplus_{S \in X} \Lambda$.
(2) $X(i)$ denotes the set of all subsets of $\{0, 1, \cdots, n\}$ of order i. Define the degree of $\Lambda^{X(0)}$ to be 0.
(3) For $S \in X(i)$ and $T \in X(i+1)$, the (S,T)-component $d_i(S,T)$ of d_i : $\Lambda^{X(i+1)} \to \Lambda^{X(i)}$ is defined as follows.

(A) If $S \not\subset T$, $d_i(S,T)(x) = 0$.

(B) If $S \subset T$, put $\{j\} = T \backslash S$.

∘ If $j = 0$, $d_i(S,T)(x) = x\tau_S$.

∘ If $j \neq 0$, $d_i(S,T)(x) = (-1)^{a(S,j)} x \omega_j$ where $a(S,j) = \sharp\{y \in S; y \leq j\}$.

Example 4.2. In the case of $[k : k^p] = p$, we have

$$0 \longrightarrow \Lambda \xrightarrow{d_1} \Lambda^{\oplus 2} \xrightarrow{d_0} \Lambda \longrightarrow 0$$

Here, $d_0(f, g) = (f\tau + g\omega_1)$ and $d_1(f) = (-f\omega_1, f\tau_{\{1\}})$.

Lemma 4.3. *The natural morphism*

$$\varprojlim_m \mathbb{Z}_p[\Gamma_{K'}]/(\Gamma_{K'})^{p^m} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\omega_1, \ldots, \omega_n]] \to \Lambda = \mathbb{Z}_p[[\Gamma_K]]$$

is an isomorphism of left $\mathbb{Z}_p[[\Gamma_{K'}]]$- and right $\mathbb{Z}_p[[\omega_1, \ldots, \omega_n]]$-modules.

Proof. For $m \in \mathbb{N}_{>0}$, put $\Gamma_m = \Gamma_{K'}/(\Gamma_{K'})^{p^m} \ltimes (\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}$. Note that the action of $\Gamma_{K'}$ on $(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}$ factors through the quotient $\Gamma_{K'}/(\Gamma_{K'})^{p^m}$. Then, we have $\mathbb{Z}_p[[\Gamma_K]] \simeq \varprojlim_m \mathbb{Z}_p[\Gamma_m]$. The natural homomorphisms of rings f : $\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \to \mathbb{Z}_p[\Gamma_m]$ and $g : \mathbb{Z}_p[(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}] \to \mathbb{Z}_p[\Gamma_m]$ induce the surjection of left $\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}]$- and right $\mathbb{Z}_p[(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}]$-modules

$$\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}] \twoheadrightarrow \mathbb{Z}_p[\Gamma_m] : a \otimes b \mapsto f(a)g(b).$$

Since both sides have the same $\mathbb{Z}_p$-rank, it turns out to be an isomorphism. On the other hand, we have

$$\begin{aligned}\varprojlim_m(\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}]) &\simeq \varprojlim_m(\varprojlim_{m'}(\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(\mathbb{Z}/p^{m'}\mathbb{Z})^{\oplus n}]))\\ &\simeq \varprojlim_m(\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}]) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\omega_1, \ldots, \omega_n]].\end{aligned}$$

This completes the proof. □

Proposition 4.4. *The sequence*

$$0 \longrightarrow \Lambda^{X(n+1)} \xrightarrow{d_n} \Lambda^{X(n)} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} \Lambda^{X(0)} \xrightarrow{Aug} \mathbb{Z}_p \longrightarrow 0$$

gives a free resolution of the left Λ-module $\mathbb{Z}_p$. Here, $\mathbb{Z}_p$ is equipped with the structure of left Λ-modules induced from the trivial action of Γ_K.

Proof. Consider the following sequence C_ω

$$0 \longrightarrow \Lambda^{Y(n)} \xrightarrow{d'_{n-1}} \Lambda^{Y(n-1)} \xrightarrow{d'_{n-2}} \cdots \xrightarrow{d'_0} \Lambda^{Y(0)} \xrightarrow{Aug} \mathbb{Z}_p[[\tau]] \longrightarrow 0.$$

Here

(1) $Y(i)$ denotes the set of all subsets of $\{1, \cdots, n\}$ of order i. (Recall that $X(i)$ denotes the set of all subsets of $\{0, 1, \cdots, n\}$.)
(2) For $S \in Y(i)$ and $T \in Y(i+1)$, the (S,T)-component $d'_i(S,T)$ of d'_i : $\Lambda^{Y(i+1)} \to \Lambda^{Y(i)}$ is defined as follows.

   ◦ If $S \not\subset T$, $d'_i(S,T)(x) = 0$.

   ◦ If $S \subset T$, put $\{j\} = T \backslash S$.
   $d'_i(S,T)(x) = (-1)^{a(S,j)} x\omega_j$ where $a(S,j) = \sharp\{y \in S; y \leq j\}$.

Put $\Lambda_0 = \mathbb{Z}_p[[\omega_1, \ldots, \omega_n]]$. Let $K.(\omega_1, \ldots, \omega_n)$ be the Koszul complex

$$0 \longrightarrow \Lambda_0^{Y(n)} \xrightarrow{d'_{n-1}} \Lambda_0^{Y(n-1)} \xrightarrow{d'_{n-2}} \cdots \xrightarrow{d'_0} \Lambda_0^{Y(0)} \longrightarrow 0.$$

Since we have the isomorphism $\varprojlim_m \mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\omega_1, \ldots, \omega_n]] \simeq \Lambda = \mathbb{Z}_p[[\Gamma_K]]$, the sequence

$$0 \longrightarrow \Lambda^{Y(n)} \xrightarrow{d'_{n-1}} \Lambda^{Y(n-1)} \xrightarrow{d'_{n-2}} \cdots \xrightarrow{d'_0} \Lambda^{Y(0)} \longrightarrow 0$$

is the complex $\varprojlim_m \mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \otimes_{\mathbb{Z}_p} K.(\omega_1, \ldots, \omega_n)$, so the sequence C_ω is a resolution of $\mathbb{Z}_p[[\Gamma_{K'}]] = \Lambda/(\sum_{i=1}^n \Lambda\omega_i)$ in the category of left Λ-modules (note that the transition map $\mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^m}] \to \mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^{m'}}]$ $(m \geq m')$ is surjective). In particular, the sequence C_ω is exact. Then, consider the following commutative diagram of left Λ-modules (the commutativity follows from the relation (4)):

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & 0 & & \\
 & & & & & & & & & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{Y(n)} & \xrightarrow{d'_{n-1}} & \Lambda^{Y(n-1)} & \xrightarrow{d'_{n-2}} & \cdots & \xrightarrow{d'_0} & \Lambda^{Y(0)} & \longrightarrow & \mathbb{Z}_p[[\tau]] & \longrightarrow & 0 \\
 & & {\scriptstyle d''_n}\downarrow & & {\scriptstyle d''_{n-1}}\downarrow & & & & {\scriptstyle d''_0}\downarrow & & {\scriptstyle \tau}\downarrow & & \\
0 & \longrightarrow & \Lambda^{Y(n)} & \xrightarrow{d'_{n-1}} & \Lambda^{Y(n-1)} & \xrightarrow{d'_{n-2}} & \cdots & \xrightarrow{d'_0} & \Lambda^{Y(0)} & \longrightarrow & \mathbb{Z}_p[[\tau]] & \longrightarrow & 0 \\
 & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & \mathbb{Z}_p & & \\
 & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & 0 & & .
\end{array}
$$

For $S \in Y(i)$ (the target) and $T \in Y(i)$ (the source), the (S,T)-component of $d''_i(S,T) : \Lambda^{Y(i)} \to \Lambda^{Y(i)}$ is defined as follows.

- ∘ If $S \neq T$, $d''_i(S,T)(x) = 0$.
- ∘ If $S = T$, $d''_i(S,T)(x) = x\tau_S$.

Since C_Λ is the simple complex associated to the mapping cone of

$$d'' : \varprojlim_n \mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^n}] \otimes_{\mathbb{Z}_p} K.(\omega_1, \ldots, \omega_n) \to \varprojlim_n \mathbb{Z}_p[\Gamma_{K'}/(\Gamma_{K'})^{p^n}] \otimes_{\mathbb{Z}_p} K.(\omega_1, \ldots, \omega_n),$$

it is quasi-isomorphic to the complex $\mathbb{Z}_p[[\tau]] \to \mathbb{Z}_p[[\tau]] : x \mapsto x\tau$, and hence to $\mathbb{Z}_p$. Thus, we get the exact sequence

$$0 \longrightarrow \Lambda^{X(n+1)} \xrightarrow{d_n} \Lambda^{X(n)} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} \Lambda^{X(0)} \xrightarrow{Aug} \mathbb{Z}_p \longrightarrow 0.$$

□

5. Proof of the main theorem

5.1. **Connection between $C_{\Gamma_K}(M)$ and C_Λ.** First, let us fix some notations. Let G denote a profinite group and put $\Lambda_G = \mathbb{Z}_p[[G]]$. Then, Λ_G-Mod (resp. $\mathbb{Z}_p$-Mod, $\mathcal{C}_G$, $\mathcal{D}_G$) denotes the category of left Λ_G-modules (resp. $\mathbb{Z}_p$-modules, compact left Λ_G-modules, discrete left Λ_G-modules). Furthermore, let $D^+(*)$ denote the derived category of $*$ ($\in \{\Lambda_G\text{-Mod}, \mathbb{Z}_p\text{-Mod}, \mathcal{C}_G, \mathcal{D}_G\}$) which consists of complexes bounded below.

Let M be a left Λ-module. Define the complex $C_{\Gamma_K}(M)$ to be

$$C_{\Gamma_K}(M) = \mathrm{Hom}_\Lambda(C_\Lambda, M)$$

where $\mathrm{Hom}_\Lambda(A,B)$ $(A,B \in \Lambda(=\Lambda_{\Gamma_K})\text{-Mod})$ denotes the set of all homomorphisms $f : A \to B$ of Λ-modules. In the case $M = D(V)$, this $C_{\Gamma_K}(M)$ clearly coincides with the one defined in Section 3. On the other hand, by Proposition 4.4, we have

$$\mathrm{Hom}_\Lambda(C_\Lambda, M) \simeq \mathrm{RHom}_\Lambda(\mathbb{Z}_p, M)$$

where we denote $\mathrm{RHom}_\Lambda(\mathbb{Z}_p, -) : D^+(\Lambda\text{-Mod}) \to D^+(\mathbb{Z}_p\text{-Mod})$.

For every discrete left Λ-module M, consider the $\mathbb{Z}_p$-module $\mathrm{Hom}_{\Lambda,cont}(\mathbb{Z}_p, M)$ of all continuous homomorphisms $f : \mathbb{Z}_p \to M$ of Λ-modules. Then, we obtain the functor

$$\mathrm{Hom}_{\Lambda,cont}(\mathbb{Z}_p, -) : \mathcal{D}_{\Gamma_K} \to \mathcal{D}_{\mathbb{Z}_p}.$$

Here, $\mathcal{D}_{\mathbb{Z}_p}$ denotes the category $\mathcal{D}_{\{e\}}$ (e: unit). To define the derived functor $\mathrm{RHom}_{\Lambda,cont}(\mathbb{Z}_p, M)$ (M: discrete left Λ-module), we can use the projective resolution of $\mathbb{Z}_p$ in $\mathcal{C}_{\Gamma_K}$ (see Remark 5.2 below). Since each component of C_Λ is a finitely generated free Λ-module, it gives a projective resolution of $\mathbb{Z}_p$ in $\mathcal{C}_{\Gamma_K}$. Furthermore, since we have the equality $\mathrm{Hom}_\Lambda(P, M) = \mathrm{Hom}_{\Lambda,cont}(P, M)$ for a finitely generated free Λ-module P and a discrete Λ-module M, we obtain

$$\mathrm{RHom}_\Lambda(\mathbb{Z}_p, M) = \mathrm{RHom}_{\Lambda,cont}(\mathbb{Z}_p, M).$$

If M is a discrete Λ-module, we also have

$$\mathrm{RHom}_{\Lambda,cont}(\mathbb{Z}_p, M) \simeq \mathrm{R}\Gamma(\Gamma_K, M)$$

(see [NSW, p231, (5.2.7)]). Thus, we obtain the following.

Proposition 5.1. *If M is a discrete left Λ-module, we have*

$$C_{\Gamma_K}(M) \simeq \mathrm{R}\Gamma(\Gamma_K, M).$$

Remark 5.2. Though it is stated in ([NSW], p231) that, to define $\mathrm{RHom}_{\Lambda_G,cont}(L, M)$ for $L \in \mathcal{C}_G$ and $M \in \mathcal{D}_G$, one can use either projective resolutions of L in $\mathcal{C}_G$ or injective resolutions of M in $\mathcal{D}_G$, we shall review this fact here. For a projective resolution $P^\cdot \to L$ in $\mathcal{C}_G$ and an injective resolution $M \to I^\cdot$ in $\mathcal{D}_G$, it suffices to show

$$\mathrm{Hom}^\cdot_{\Lambda_G,cont}(L, I^\cdot) \to \mathrm{Hom}^\cdot_{\Lambda_G,cont}(P^\cdot, I^\cdot) \leftarrow \mathrm{Hom}^\cdot_{\Lambda_G,cont}(P^\cdot, M)$$

are quasi-isomorphisms. Here, $\mathrm{Hom}_{\Lambda_G,cont}(A, B)$ ($A \in \mathcal{C}_G$ and $B \in \mathcal{D}_G$) denotes all continuous homomorphisms $f : A \to B$ of Λ_G-modules. For this, we have to show that both functors $\mathcal{C}_G \to \mathcal{D}_{\mathbb{Z}_p} : L \mapsto \mathrm{Hom}_{\Lambda_G,cont}(L, I)$ (I is an injective object of $\mathcal{D}_G$) and $\mathcal{D}_G \to \mathcal{D}_{\mathbb{Z}_p} : M \mapsto \mathrm{Hom}_{\Lambda_G,cont}(P, M)$ (P is a projective object of $\mathcal{C}_G$) are exact functors. This follows from the fact that, for $L \in \mathcal{C}_G$ and $M \in \mathcal{D}_G$, any continuous homomorphism $L \to M$ of Λ_G-modules factors through a compact and discrete subgroup of M .

Remark 5.3. The functor C_{Γ_K} from the category Λ-Mod (resp. $\mathcal{D}_{\Gamma_K}$) to the category $\mathbb{Z}_p$-Mod (resp. $\mathcal{D}_{\mathbb{Z}_p}$) naturally extends to the functor C_{Γ_K} from the derived category $D^+(\Lambda\text{-Mod})$ (resp. $D^+(\mathcal{D}_{\Gamma_K})$) to the derived category $D^+(\mathbb{Z}_p$-Mod) (resp. $D^+(\mathcal{D}_{Z_p})$). Note that the functor C_{Γ_K} is an exact functor, i.e. for an exact sequence of Λ-modules (resp. discrete Λ-modules) $0 \to M_1 \to M_2 \to$

$M_3 \to 0$, we have an exact sequence of complexes $0 \to C_{\Gamma_K}(M_1) \to C_{\Gamma_K}(M_2) \to C_{\Gamma_K}(M_3) \to 0$. Furthermore, Proposition 5.1 induces a canonical isomorphism of functors $C_{\Gamma_K}(-) \simeq \mathrm{R}\Gamma(\Gamma_K, -)$ from the derived category $D^+(\mathcal{D}_{\Gamma_K})$ to the derived category $D^+(\mathcal{D}_{\mathbb{Z}_p})$.

The exact functor from the category $\mathcal{D}_{\Gamma_K}$ to the category Λ-Mod naturally extends to the functor from the derived category $D^+(\mathcal{D}_{\Gamma_K})$ to the derived category $D^+(\Lambda\text{-Mod})$. Therefore, the object $\mathrm{R}\Gamma(H_K, V)$ of the derived category $D^+(\mathcal{D}_{\Gamma_K})$ gives an object of the derived category $D^+(\Lambda\text{-Mod})$.

Proposition 5.4. *Let V be a p-torsion representation of G_K. Then, we have an isomorphism*

$$\mathrm{R}\Gamma(H_K, V) \simeq [D(V) \overset{\rho=\phi-1}{\to} D(V)]$$

in $D^+(\Lambda\text{-Mod})$.

For the proof of this proposition, we shall introduce a subcategory of Λ_{G_K}-Mod which contains the Λ_{G_K}-module $\mathbb{A} \otimes_{\mathbb{Z}_p} V$. First, let us fix some notations. Let G be a profinite group and H be a closed normal subgroup of G. Let $\mathcal{S}$ denote the set of open subgroups of H which are also normal subgroups of G. We define $\mathcal{E}_{G,H}$ to be the full subcategory of Λ_G-Mod which consists of Λ_G-modules M with the following property: for all $x \in M$, there exist $U_x \in \mathcal{S}$ and $n_x \in \mathbb{Z}_{>0}$ such that the action of $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/U_x}/p^{n_x})$ on x is 0. Then, $\mathcal{E}_{G,H}$ forms an abelian category.

Lemma 5.5. *The category $\mathcal{E}_{G,H}$ has sufficiently many injectives.*

Proof. For $M \in \mathcal{E}_{G,H}$, there exists an inclusion $M \hookrightarrow I$ where I is an injective object of Λ_G-Mod. Define I' to be $\{x \in I \mid \exists U \in \mathcal{S}, n \in \mathbb{Z}_{>0}$ s.t. the action of $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/U}/p^n)$ on x is 0 $\}$. Then, I' becomes an injective object of $\mathcal{E}_{G,H}$ such that $M \subset I'$. □

Lemma 5.6. (1) *For $U, U' \in \mathcal{S}$, $U' \subset U$, the homomorphism $\Lambda_{G/U'} \otimes_{\Lambda_{H/U'}} \Lambda_{H/U} \to \Lambda_{G/U}$ is an isomorphism.*
(2) *For $U \in \mathcal{S}$, $\Lambda_{G/U}$ is flat as a right $\Lambda_{H/U}$-module.*

Proof. (1) The natural homomorphism $G/U' \to G/H$ has a continuous section $s : G/H \to G/U'$ (see [S2], p4, Proposition 1.). With this, we obtain a homeomorphism $G/H \times H/U' \simeq G/U' : (a, b) \mapsto s(a) \cdot b$ of profinite sets which is compatible with the right action of H/U'. Therefore, we get an isomorphism $f' : \mathbb{Z}_p[[G/H]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[H/U'] \simeq \mathbb{Z}_p[[G/U']]$ of right $\mathbb{Z}_p[H/U']$-modules. By using the composition with the section s and $G/U' \to G/U$, we similarly get an isomorphism $f : \mathbb{Z}_p[[G/H]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[H/U] \simeq \mathbb{Z}_p[[G/U]]$ of right $\mathbb{Z}_p[H/U]$-modules. Since f and f' are compatible with $\mathbb{Z}_p[H/U'] \to \mathbb{Z}_p[H/U]$ and $\mathbb{Z}_p[[G/U']] \to \mathbb{Z}_p[[G/U]]$, we obtain the desired result.

(2) Since we have the isomorphism $f : \mathbb{Z}_p[[G/H]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[H/U] \simeq \mathbb{Z}_p[[G/U]]$ of right $\mathbb{Z}_p[H/U]$-modules and $\mathbb{Z}_p[[G/H]]$ is flat as a $\mathbb{Z}_p$-module, $\Lambda_{G/U}$ is flat as a right $\Lambda_{H/U}$-module. □

For $M \in \mathcal{D}_H$ and $U \in \mathcal{S}$, define $M^U = \{x \in M \,|\, \text{the action of } \mathrm{Ker}\,(\Lambda_H \to \Lambda_{H/U}) \text{ on } x \text{ is trivial}\}$. Since M is an object of $\mathcal{D}_H$, we have $M = \varinjlim_{U\in\mathcal{S}} M^U$. Define the left $\Lambda_{G/U}$-module $T_U(M)$ to be $\Lambda_{G/U} \otimes_{\Lambda_{H/U}} M^U$. By Lemma 5.6.(1), for $U' \in \mathcal{S}$, $U' \subset U$, the natural morphism $\Lambda_{G/U'} \otimes_{\Lambda_{H/U'}} M^U \to T_U(M)$ becomes an isomorphism. Therefore, by Lemma 5.6.(2), we obtain an injection $T_U(M) \to T_{U'}(M)$ which is compatible with the action of Λ_G. Then, it follows easily that $\{\Lambda_U(M) \,|\, U \in \mathcal{S}\}$ forms an inductive system. We denote the inductive limit $\varinjlim_{U\in\mathcal{S}} T_U(M)$ by $T(M)$. Since $T(M)$ becomes an object of $\mathcal{E}_{G,H}$, we obtain a functor $T : \mathcal{D}_H \to \mathcal{E}_{G,H}$. Furthermore, by Lemma 5.6.(2) and the fact $M = \varinjlim_{U\in\mathcal{S}} M^U$, it follows that the functor T is an exact functor.

Lemma 5.7. *If H is a finite group,* $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/H})$ *is generated by* $\{h-1 \,|\, h \in H\}$.

Proof. There exists an exact sequence of projective systems of finite abelian groups

$$\bigoplus_{h\in H} \mathbb{Z}/p^n[G/V] \cdot (h-1) \to \mathbb{Z}/p^n[G/V] \to \mathbb{Z}/p^n[G/(V\cdot H)] \to 0$$

where n and V run through positive integers and open normal subgroups of G. Since these are projective systems of finite abelian groups, the filtered projective limit preserves the exactness by Pontryagin duality. Thus, we obtain an exact sequence

$$\bigoplus_{h\in H} \Lambda_G \cdot (h-1) \to \Lambda_G \to \Lambda_{G/H} \to 0.$$

□

Lemma 5.8. *Let N be an object of $\mathcal{E}_{G,H}$. If the action of* $\mathrm{Ker}\,(\Lambda_H \to \Lambda_{H/U})$ *on x is 0 for $x \in N, U \in \mathcal{S}$, then, the action of* $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/U})$ *on x is also 0.*

Proof. By the definition of $\mathcal{E}_{G,H}$, there exists an element $U' \in \mathcal{S}$ contained in U such that the action of $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/U'})$ on x is 0. By applying Lemma 5.7 above to $U/U' \subset G/U'$, we see that $\mathrm{Ker}\,(\Lambda_{G/U'} \to \Lambda_{G/U})$ is an ideal generated by $\{g-1 \,|\, g \in U/U'\}$. Since the action of U on x is trivial by hypothesis, the action of this ideal on x is 0. □

Proposition 5.9. *The functor T is a left-adjoint functor of the forgetful functor $F : \mathcal{E}_{G,H} \to \mathcal{D}_H$.*

Proof. For an object M of $\mathcal{D}_H$, the natural map $M^U \to T_U(M) : x \mapsto 1 \otimes x$ is a homomorphism of Λ_H-modules and compatible with respect to U. By taking the inductive limit, we obtain $\alpha_M : M \to F \circ T(M)$. This morphism is functorial in

M. On the other hand, for an object N of $\mathcal{E}_{G,H}$, N^U becomes a $\Lambda_{G/U}$-module by Lemma 5.8 above. Therefore, we have a homomorphism $T_U(N) \to N^U$ of $\Lambda_{G/U}$-modules and this homomorphism is compatible with respect to U. By taking the inductive limit, we obtain $\beta_N : T \circ F(N) \to N$. This morphism is functorial in N. For $M \in \mathcal{D}_H$ and $N \in \mathcal{E}_{G,H}$, we obtain maps which are functorial in M and N

$$\mathrm{Hom}_{\mathcal{E}_{G,H}}(T(M), N) \to \mathrm{Hom}_{\mathcal{D}_H}(M, F(N)) : \varphi \mapsto F(\varphi) \circ \alpha_M,$$
$$\mathrm{Hom}_{\mathcal{D}_H}(M, F(N)) \to \mathrm{Hom}_{\mathcal{E}_{G,H}}(T(M), N) : \psi \mapsto \beta_N \circ T(\psi).$$

It follows easily that each map is inverse to the other map. □

Since the functor T is exact and a left-adjoint functor of F by Proposition 5.9, the functor F preserves injective objects.

Now, for an object N of $\mathcal{E}_{G,H}$, define $N^H = \{x \in N \,|\, h(x) = x, \forall h \in H\}$.

Lemma 5.10. *N^H is a left $\Lambda_{G/H}$-module*

Proof. For $x \in N^H$, there exists an element $U \in \mathcal{S}$ such that the action of $\mathrm{Ker}\,(\Lambda_G \to \Lambda_{G/U})$ on x is 0. By applying Lemma 5.7 to $H/U \subset G/U$, it follows that the ideal $\mathrm{Ker}\,(\Lambda_{G/U} \to \Lambda_{G/H})$ is generated by $\{h - 1 \,|\, h \in H/U\}$. Thus, we see that the action of this kernel on x is 0. □

With this, we have a left exact functor $\Gamma_{\mathcal{E}}(H, -) : \mathcal{E}_{G,H} \to \Lambda_{G/H}\text{-Mod} : N \mapsto N^H$ and

$$\mathrm{R}\Gamma_{\mathcal{E}}(H, -) : D^+(\mathcal{E}_{G,H}) \to D^+(\Lambda_{G/H}\text{-Mod}).$$

Proposition 5.11. *The following diagram is commutative*

$$\begin{array}{ccc} D^+(\mathcal{E}_{G,H}) & \xrightarrow{\mathrm{R}\Gamma_{\mathcal{E}}(H,-)} & D^+(\Lambda_{G/H}\text{-}Mod) \\ {\scriptstyle F_1}\downarrow & & {\scriptstyle F_2}\downarrow \\ D^+(\mathcal{D}_H) & \xrightarrow{\mathrm{R}\Gamma(H,-)} & D^+(\mathbb{Z}_p\text{-}Mod). \end{array}$$

Here the two vertical arrows denote the functors induced by the forgetful functors $\mathcal{E}_{G,H} \to \mathcal{D}_H$ and $\Lambda_{G/H}$-Mod $\to \mathbb{Z}_p$-Mod.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{G,H} & \xrightarrow{\Gamma_{\mathcal{E}}(H,-)} & \Lambda_{G/H}\text{-Mod} \\ \downarrow & & \downarrow \\ \mathcal{D}_H & \xrightarrow{\Gamma(H,-)} & \mathbb{Z}_p\text{-Mod}. \end{array}$$

The two vertical functors are exact and the left vertical map preserves injective objects by Proposition 5.9. Thus, it follows easily that the diagram in this proposition is commutative. □

Proposition 5.12. *Let F_3 (resp. F_4) be the functor $D^+(\mathcal{D}_G) \to D^+(\mathcal{E}_{G,H})$ (resp. $D^+(\mathcal{D}_{G/H}) \to D^+(\Lambda_{G/H}\text{-}Mod)$) induced by the inclusion functor $\mathcal{D}_G \to \mathcal{E}_{G,H}$ (resp. $\mathcal{D}_{G/H} \to \Lambda_{G/H}\text{-}Mod$). Then, the following diagram is commutative*

$$\begin{array}{ccc} D^+(\mathcal{D}_G) & \xrightarrow{\mathrm{R}\Gamma(H,-)} & D^+(\mathcal{D}_{G/H}) \\ {\scriptstyle F_3}\downarrow & & {\scriptstyle F_4}\downarrow \\ D^+(\mathcal{E}_{G/H}) & \xrightarrow{\mathrm{R}\Gamma_{\mathcal{E}}(H,-)} & D^+(\Lambda_{G/H}\text{-}Mod). \end{array}$$

Proof. It suffices to show that, for an injective object I of $\mathcal{D}_G$, we have $\mathrm{R}^i\Gamma_{\mathcal{E}}(H, F_3(I)) = 0$ $(i > 0)$. By Proposition 5.11, we have an isomorphism $\mathrm{R}^i\Gamma_{\mathcal{E}}(H, F_3(I)) = \mathrm{R}^i\Gamma(H, F_1 \circ F_3(I))$ of $\mathbb{Z}_p$-modules. Since the following diagram is commutative by the group cohomology theory for discrete modules, we obtain $\mathrm{R}^i\Gamma(H, F_1 \circ F_3(I)) = F_2 \circ F_4(\mathrm{R}^i\Gamma(H, I)) = 0$.

$$\begin{array}{ccc} D^+(\mathcal{D}_G) & \xrightarrow{\mathrm{R}\Gamma(H,-)} & D^+(\mathcal{D}_{G/H}) \\ {\scriptstyle F_1\circ F_3}\downarrow & & {\scriptstyle F_2\circ F_4}\downarrow \\ D^+(\mathcal{D}_H) & \xrightarrow{\mathrm{R}\Gamma(H,-)} & D^+(\mathbb{Z}_p\text{-Mod}). \end{array}$$

□

Now, we shall give the proof of Proposition 5.4. Note that, since $\mathbb{A} \otimes_{\mathbb{Z}_p} V$ becomes an object of $\mathcal{E}_{G_K,H_K}$ (see Remark 2.6), we have an exact sequence

$$0 \to V \to \mathbb{A} \otimes_{\mathbb{Z}_p} V \overset{\rho=\phi-1}{\to} \mathbb{A} \otimes_{\mathbb{Z}_p} V \to 0$$

in $\mathcal{E}_{G_K,H_K}$. First, we will show that we have

$$H^i(H_K, \mathbb{A} \otimes_{\mathbb{Z}_p} V) = 0 \quad \text{for all } i > 0.$$

Since we have the canonical isomorphism of Galois groups $H_K \simeq G_{\mathbb{E}_K}$ by the theory of field of norms, we have only to show $H^i(G_{\mathbb{E}_K}, \mathbb{A}\otimes_{\mathbb{Z}_p} V) = 0$ for all $i > 0$. On the other hand, we have isomorphisms of $G_{\mathbb{E}_K}$ $(\simeq H_K)$-modules $\mathbb{A} \otimes_{\mathbb{Z}_p} V \simeq \mathbb{A} \otimes_{\mathbb{A}_K} D(V) \simeq \bigoplus_{j=1}^d \mathbb{A}/p^{m_j}\mathbb{A}$. Thus, it suffices to show $H^i(G_{\mathbb{E}_K}, \mathbb{A}/p^m\mathbb{A}) = 0$ for all $i > 0$. This is clear for $m = 1$ ($H^i(G_{\mathbb{E}_K}, \mathbb{E}) = 0$ for all $i > 0$) and the general case can be deduced by induction on the integer m. Thus, by using Proposition 5.11, we obtain isomorphisms in $D^+(\Lambda\text{-Mod})$ from the exact sequence above

$$\begin{aligned} \mathrm{R}\Gamma_{\mathcal{E}}(H_K, V) \simeq& \mathrm{R}\Gamma_{\mathcal{E}}(H_K, [\mathbb{A} \otimes_{\mathbb{Z}_p} V \overset{\phi-1}{\to} \mathbb{A} \otimes_{\mathbb{Z}_p} V]) \\ \simeq& \Gamma_{\mathcal{E}}(H_K, [\mathbb{A} \otimes_{\mathbb{Z}_p} V \overset{\phi-1}{\to} \mathbb{A} \otimes_{\mathbb{Z}_p} V]) \\ =& [D(V) \overset{\phi-1}{\to} D(V)]. \end{aligned}$$

On the other hand, by Proposition 5.12, $\mathrm{R}\Gamma_{\mathcal{E}}(H_K, V)$ coincides with the image of the Galois cohomology $\mathrm{R}\Gamma(H_K, V) \in D^+(\mathcal{D}_{\Gamma_K})$ by the functor $F_4 : D^+(\mathcal{D}_{\Gamma_K}) \to D^+(\Lambda\text{-Mod})$. Thus, this completes the proof of Proposition 5.4.

5.2. **Conclusion.** We now compute the Galois cohomology $\mathrm{R}\Gamma(G_K, V)$ for a p-torsion representation of V of G_K. We have

$$\mathrm{R}\Gamma(G_K, V) \simeq \mathrm{R}\Gamma(\Gamma_K, \mathrm{R}\Gamma(H_K, V)).$$

From Proposition 5.1 and Remark 5.3, we obtain

$$\mathrm{R}\Gamma(\Gamma_K, \mathrm{R}\Gamma(H_K, V)) \simeq C_{\Gamma_K}(\mathrm{R}\Gamma(H_K, V)).$$

By Proposition 5.4,

$$\mathrm{R}\Gamma(G_K, V) \simeq C_{\Gamma_K}([D(V) \xrightarrow{\rho} D(V)]) \simeq C_{\phi,\Gamma_K}(D(V)).$$

Thus, this completes the proof of the main theorem.

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References

[A] Andreatta, F.: Generalized ring of norms and generalized (ϕ, Γ)-modules. Preprint.

[Bn] Benois, D.: On Iwasawa theory of crystalline representations. Duke. Math. 104 (2000), 211–267.

[Br1] Berger, L.: Représentations p-adiques et équations différentielles. Invent. Math. 148 (2002), 219–284.

[Br2] Berger, L.: An introduction to the theory of p-adic representations. Geometric aspects of Dwork theory. Vol. I, II, 255–292, 2004.

[F] Fontaine, J-M.: Représentations p-adiques des corps locaux. I. The Grothendieck Festschrift, Vol. II, 249–309, Progr. Math., 87, Birkhauser, 1990.

[FW] Fontaine, J-M., Wintenberger, J-P.: Le "corps des normes" de certaines extensions algébriques de corps locaux. C. R. Acad. Sci. Paris Ser. A-B 288 (1979), A367–A370.

[H1] Herr, L.: Sur la cohomologie galoisienne des corps p-adiques. Bull. Soc. Math. France 126 (1998), 563–600.

[H2] Herr, L.: Φ-Γ-modules and Galois cohomology. Invitation to higher local fields (Munster, 1999), 263–272 (electronic), Geom. Topol. Monogr., 3, Geom. Topol. Publ., Coventry, 2000.

[H3] Herr, L.: Une approche nouvelle de la dualité locale de Tate. Math. Ann. 320 (2001), 307–337.

[NSW] Neukirch, J., Schmidt, A., Wingberg, K.: Cohomology of number fields. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 323. Springer-Verlag, Berlin, 2000.

[S1] Serre, J-P.: Local fields. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979.

[S2] Serre, J-P.: Galois cohomology. Springer-Verlag, Berlin, 1997

[W] Wintenberger, J-P.: Le corps des normes de certaines extensions infinies de corps locaux; applications. Ann. Sci. École Norm. Sup. (4) 16 (1983), 59–89.

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