GENERALIZATION OF THE THEORY OF SEN IN THE SEMI-STABLE REPRESENTATION CASE

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Abstract. For a semi-stable representation V, we will construct a subspace $D_{\pi\text{-Sen}}(V)$ of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ endowed with a linear derivation $\nabla^{(\pi)}$. The action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is closely related to the action of the monodromy operator N on $D_{\mathrm{st}}(V)$. Furthermore, in the geometric case, the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.

1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. By a p-adic representation of $G_K = \mathrm{Gal}(\overline{K}/K)$, we mean a finite dimensional vector space V over $\mathbb{Q}_p$ endowed with a continuous action of G_K. Put $K_\infty = \cup_{0\leq m} K(\zeta_{p^m})$ where ζ_{p^m} denote a primitive p^m-th root of unity in $\overline{K}$ satisfying $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$. Let H_K denote the kernel of the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^*$ and define Γ_K to be $G_K/H_K \simeq \mathrm{Gal}(K_\infty/K)$. Then, for a p-adic representation V of G_K, Sen constructs a K_∞-vector space $D_{\mathrm{Sen}}(V)$ of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ equipped with the K_∞-linear derivation $\nabla^{(0)}$ which is the p-adic Lie algebra of Γ_K. In the case when V is a Hodge-Tate representation of G_K, the set of eigenvalues of $\nabla^{(0)}$ on $D_{\mathrm{Sen}}(V)$ is exactly the same as the set of Hodge-Tate weights of V.

Now, we shall state the aim of this article. First, let us fix some notations. Fix a prime π of $\mathcal{O}_K$ (the ring of integers of K) and for each $1 \leq m$, fix a p^m-th root π^{1/p^m} of π in $\overline{K}$ satisfying $(\pi^{1/p^{m+1}})^p = \pi^{1/p^m}$. Put $K^{\mathrm{BK}} = \cup_{0\leq m} K(\pi^{1/p^m})$ and $K_\infty^{\mathrm{BK}} = \cup_{0\leq m} K_\infty(\pi^{1/p^m})$. Here, the letter BK stands for the Breuil-Kisin extension. Let H denote the Galois group $\mathrm{Gal}(\overline{K}/K_\infty^{\mathrm{BK}})$ and define Γ_{BK} to be $\mathrm{Gal}(K_\infty^{\mathrm{BK}}/K_\infty)$. In this article, for a semi-stable representation V of G_K, we shall construct a K_∞^{BK}-vector space $D_{\pi\text{-Sen}}(V)$ of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ equipped with the K_∞^{BK}-linear derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$. Here, $\nabla^{(\pi)}$

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denotes the p-adic Lie algebra of Γ_{BK}. Then, the action of $\nabla^{(0)}$ on $D_{\pi\text{-Sen}}(V)$ tells us about Hodge-Tate weights as in the case of $D_{\mathrm{Sen}}(V)$ and the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is closely related to the action of the monodromy operator N on $D_{\mathrm{st}}(V)$. Furthermore, in the case $V = H^m_{\text{ét}}(X \otimes_K \overline{K}, \mathbb{Q}_p)$ where X denotes a proper smooth scheme over K, the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.

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2. Preliminaries on basic facts

2.1. **p-adic periods rings and p-adic representations.** (See [F1] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Put $K_0 = \mathrm{Frac}(W(k))$ where $W(k)$ denotes the ring of Witt vectors with coefficients in k. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, ...) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}.$$

For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, define their sum and product by $(x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. Let $\epsilon = (\epsilon^{(n)})$ denote an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is a perfect field of characteristic $p > 0$ and is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widetilde{\mathbb{E}}$ is equipped with an action of a Frobenius σ and a continuous action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Define $\widetilde{\mathbb{E}}^+$ to be the ring of integers for this valuation. Put $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k \gg -\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$ where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^+$. This ring $\widetilde{\mathbb{B}}^+$ is equipped with a surjective homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p : \quad \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

Let $\widetilde{p}$ denote $(p^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$. Then, $\mathrm{Ker}\,(\theta)$ is the principal ideal generated by $\omega = [\widetilde{p}] - p$. The ring B^+_{dR} is defined to be the $\mathrm{Ker}\,(\theta)$-adic completion of $\widetilde{\mathbb{B}}^+$

$$B^+_{\mathrm{dR}} = \varprojlim_{n \geq 0} \widetilde{\mathbb{B}}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in B^+_{dR} is a generator of the maximal ideal. Put $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[1/t]$. The ring B_{dR} becomes a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$ ($i \in \mathbb{Z}$). Then, $(B_{\mathrm{dR}})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR}}(V)).$$

Define B_{HT} to be the associated graded algebra to the filtration $\mathrm{Fil}^i B_{\mathrm{dR}}$. The quotient $\mathrm{gr}^i B_{\mathrm{HT}} = \mathrm{Fil}^i B_{\mathrm{dR}} / \mathrm{Fil}^{i+1} B_{\mathrm{dR}}$ ($i \in \mathbb{Z}$) is a one-dimensional $\mathbb{C}_p$-vector space spanned by the image of t^i. Thus, we obtain the presentation

$$B_{\mathrm{HT}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$$

where $\mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i)$ is the Tate twist. Then, $(B_{\mathrm{HT}})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT}}(V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT}}(V)).$$

Let $\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ be the natural homomorphism where $\mathcal{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$. Define the ring A_{cris} to be the p-adic completion of the PD-envelope of $\mathrm{Ker}\,(\theta)$ compatible with the canonical PD-envelope over the ideal generated by p. Put $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ and $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t]$. These rings are K_0-algebras endowed with actions of G_K and Frobenius φ. Furthermore, since these rings are canonically included in B_{dR}, they are endowed with the filtration induced by that of B_{dR}. Then, $(B_{\mathrm{cris}})^{G_K}$ is canonically isomorphic to K_0. Thus, for a p-adic representation V of G_K, $D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space. We say that a p-adic representation V of G_K is a crystalline representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{cris}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{cris}}(V)).$$

Fix a prime element π of $\mathcal{O}_K$ (the ring of integers of K) and an element $s = (s^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $s^{(0)} = \pi$. Then, the series $\log([s]\pi^{-1})$ converges to an element u_π in B_{dR}^+ and the subring $B_{\mathrm{cris}}[u_\pi]$ of B_{dR} depends only on the choice of π. We denote this ring by B_{st}. Since this ring is included in B_{dR}, it is endowed with the action of G_K and the filtration induced by those on B_{dR}. The element u_π is transcendental over B_{cris} and we extend the Frobenius φ on B_{cris} to B_{st} by putting $\varphi(u_\pi) = p u_\pi$. Furthermore, define the B_{cris}-derivation $B_{\mathrm{st}} \to B_{\mathrm{st}}$ by $N(u_\pi) = -1$. It is easy to verify that we have $N\varphi = p\varphi N$ and that the action of N on $D_{\mathrm{st}}(V)$ is nilpotent. As in the case of B_{cris}, we have $(B_{\mathrm{st}})^{G_K} = K_0$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space. We say that a p-adic representation V of G_K is a semi-stable representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{st}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{st}}(V)).$$

Furthermore, we say that V is a potentially semi-stable representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a semi-stable representation of G_L. Due to the result of Berger [Be1], it is known that V is a potentially semi-stable representation of G_K if and only if V is a de Rham representation of G_K.

2.2. **The theory of Sen.** Keep the notation and assumption in Introduction. In the article [S3], Sen shows that, for a p-adic representation V of G_K, the $\hat{K}_\infty (= (\mathbb{C}_p)^{H_K})$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional K_∞-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_K is a K_∞-vector space of dimension d stable under Γ_K (called $D_{\mathrm{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty} D_{\mathrm{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}_\infty \otimes_{K_\infty} D_{\mathrm{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_K$ is close enough to 1, then the series of operators on $D_{\mathrm{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to a K_∞-linear derivation $\nabla^{(0)} : D_{\mathrm{Sen}}(V) \to D_{\mathrm{Sen}}(V)$ and does not depend on the choice of γ. By the following proposition, we can see that the set of eigenvalues of $\nabla^{(0)}$ on $D_{\mathrm{Sen}}(V)$ is exactly the same as the set of Hodge-Tate weights of V if V is a Hodge-Tate representation of G_K.

Proposition 2.1. *If V is a Hodge-Tate representation of G_K, there exists a Γ_K-equivariant isomorphism of K_∞-vector spaces*

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty(n_j) \quad (n_j \in \mathbb{Z}).$$

Proof. Since V is a Hodge-Tate representation of G_K, there exists a basis $\{g_j\}_{j=1}^d$ of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ over $\mathbb{C}_p$ such that it gives the Hodge-Tate decomposition

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^{d} \mathbb{C}_p(n_j) : g_j \mapsto t^{n_j} \ (n_j \in \mathbb{Z}).$$

From this presentation, it follows that $\{g_j\}_{j=1}^d$ forms a basis of a K_∞-vector space X which is contained in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ and stable under the action of Γ_K. Then, since we have $X \hookrightarrow D_{\mathrm{Sen}}(V)$ by definition and both sides have the same dimension d over K_∞, we get the equality $X = D_{\mathrm{Sen}}(V)$. Thus, we obtain the Γ_K-equivariant isomorphism of K_∞-vector spaces $D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^d K_\infty(n_j) : g_j \mapsto t^{n_j}$. □

3. Generalization of $D_{\mathrm{Sen}}(V)$

Let us recall notations. Fix a prime π of $\mathcal{O}_K$ (the ring of integers of K) and for each $1 \leq m$, fix a p^m-th root π^{1/p^m} of π in $\overline{K}$ satisfying $(\pi^{1/p^{m+1}})^p = \pi^{1/p^m}$. Put $K^{\mathrm{BK}} = \cup_{0 \leq m} K(\pi^{1/p^m})$ and $K^{\mathrm{BK}}_\infty = \cup_{0 \leq m} K_\infty(\pi^{1/p^m})$. Let H denote the Galois group $\mathrm{Gal}(\overline{K}/K^{\mathrm{BK}}_\infty)$ and define Γ_{BK} to be $\mathrm{Gal}(K^{\mathrm{BK}}_\infty/K_\infty)$. For $\beta \in \Gamma_{\mathrm{BK}}$, we have $\beta(\zeta_{p^m}) = \zeta_{p^m}$ and define the homomorphism $c : \Gamma_{\mathrm{BK}} \to \mathbb{Z}_p$ such that we have $\beta(\pi^{1/p^m}) = \pi^{1/p^m}\zeta_{p^m}^{c(\beta)}$. Then, the homomorphism c defines an isomorphism $\Gamma_{\mathrm{BK}} \simeq \mathbb{Z}_p$ of profinite groups.

3.1. **Construction of $D_{\pi\text{-Sen}}(V)$.** Let V be a semi-stable representation of G_K. For simplicity, assume that the number of the nilpotent block of the monodromy operator N on $D_{\mathrm{st}}(V)$ is 1. In the general case, one can easily construct $D_{\pi\text{-Sen}}(V)$ in the same way. Then, there exists a basis $\{g_j\}_j$ of $D_{\mathrm{st}}(V)$ over K_0 such that we have

$$g_1 \xrightarrow{N} g_2 \xrightarrow{N} \cdots \xrightarrow{N} g_d \xrightarrow{N} 0.$$

By twisting $\{g_j\}_j$ by some powers of t in $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, we obtain a basis $\{f_j\}_j$ of $B^+_{\mathrm{cris}}[u_\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B^+_{\mathrm{cris}}[u_\pi/t]$. Then, we can write

$$(*)\begin{cases} \circ\ f_1 = t^{m_1(=0)}(F_1 + (-1)^1 \frac{u_\pi^1}{1!t^1} F_2 + (-1)^2 \frac{u_\pi^2}{2!t^2} F_3 + \cdots + (-1)^{d-1} \frac{u_\pi^{d-1}}{(d-1)!t^{d-1}} F_d) \\ \circ\ f_2 = \qquad t^{m_2}(F_2 + (-1)^1 \frac{u_\pi^1}{1!t^1} F_3 + \cdots + (-1)^{d-2} \frac{u_\pi^{d-2}}{(d-2)!t^{d-2}} F_d) \\ \qquad \cdots \\ \circ\ f_d = \qquad t^{m_d} F_d \end{cases}$$

where $\{F_j\}_j$ denotes a set of elements of $B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$ and we take $m_j \in \mathbb{Z}$ such that $\{f_j\}_j$ forms a basis of $B^+_{\mathrm{cris}}[u_\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B^+_{\mathrm{cris}}[u_\pi/t]$.

Definition 3.1. With notations as above, let $\{h_j := \overline{t^{m_j} F_j}\}_j$ denote the image of $\{t^{m_j} F_j\}_j$ by the homomorphism $B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \twoheadrightarrow \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. Then, define $D_{\pi\text{-Sen}}(V)$ to be the K^{BK}_∞-vector space generated by $\{h_j\}_j$ contained in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$.

Lemma 3.2. *The elements $\{h_j\}_j$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. In particular, $\{h_j\}_j$ forms a basis of $D_{\pi\text{-Sen}}(V)$ over K^{BK}_∞ and its dimension over K^{BK}_∞ is equal to $\dim_{\mathbb{Q}_p} V$.*

Proof. We can show inductively that $\{h_d\}$, $\{h_{d-1}, h_d\}$, ..., $\{h_1, \ldots, h_d\}$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. □

By this lemma, we can easily verify that the following proposition holds.

Proposition 3.3. *(c.f. Subsection 2.2) We have $\mathbb{C}_p \otimes_{K^{\mathrm{BK}}_\infty} D_{\pi\text{-Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $(\mathbb{C}_p)^H \otimes_{K^{\mathrm{BK}}_\infty} D_{\pi\text{-Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism.*

It follows easily that the K^{BK}_∞-vector space $D_{\pi\text{-Sen}}(V)$ is equipped with the K^{BK}_∞-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_K$ is close enough to 1. By the following proposition, we can see that the action of $\nabla^{(0)}$ on $D_{\pi\text{-Sen}}(V)$ tells us about Hodge-Tate weights as in the case of $D_{\mathrm{Sen}}(V)$.

Proposition 3.4. *(c.f. Proposition 2.1) For a semi-stable representation V of G_K, there exists a Γ_K-equivariant isomorphism of K_∞^{BK}-vector spaces*

$$D_{\pi\text{-Sen}}(V) \simeq \bigoplus_{j=1}^{d} K_\infty^{\mathrm{BK}}(n_j) : h_j \mapsto t^{n_j} \quad (n_j \in \mathbb{Z}).$$

Furthermore, the set of integers $\{n_j\}_j$ is exactly the same as the set of Hodge-Tate weights of V.

Proof. Note that we have $\{\gamma(f_j) = \chi(\gamma)^{n_j} f_j\}_j$ by definition. Then, we can show inductively that we have $\{\gamma(h_d) = \chi(\gamma)^{n_d} h_d\}$, $\{\gamma(h_{d-1}) = \chi(\gamma)^{n_{d-1}} h_{d-1}\}$, $\ldots$, $\{\gamma(h_1) = \chi(\gamma)^{n_1} h_1\}$. The rest is easily verified by Proposition 3.3. □

On the other hand, if $\beta \in \Gamma_{\mathrm{BK}}$ is close enough to 1, the series of operators on $D_{\pi\text{-Sen}}(V)$

$$\nabla^{(\pi)} = \frac{\log(\beta)}{c(\beta)} = -\frac{1}{c(\beta)} \sum_{k \geq 1} \frac{(1-\beta)^k}{k}$$

converges to a K_∞^{BK}-linear derivation on $D_{\pi\text{-Sen}}(V)$ does not depend on the choice of $\beta \in \Gamma_{\mathrm{BK}}$. This easily follows from the calculations $\nabla^{(\pi)}(f_j) = 0$ and $\nabla^{(\pi)}(\frac{u_\pi}{t}) = 1$.

Remark 3.5. By using the calculations $\nabla^{(\pi)}(f_j) = 0$ and $\nabla^{(\pi)}(\frac{u_\pi}{t}) = 1$, we obtain $\nabla^{(\pi)}(F_j) = F_{j+1}$ $(j < d)$ and $\nabla^{(\pi)}(F_d) = 0$. Thus, we can rewrite $(*)$ as

$$(*)' \quad f_j = t^{m_j}\Big(\sum_{k=0}^{d-j} (-1)^k \frac{u_\pi^k}{k! t^k} (\nabla^{(\pi)})^k (F_j)\Big) \quad (1 \leq j \leq d).$$

Compare this formula to the main construction $\{f_j^{(*)}\}_j$ in [M1]. In fact, the idea of the construction of $D_{\pi\text{-Sen}}(V)$ is based on the similarity between Corollary 2.1.14 of [Ki] and Main Theorems of [M1] and [M2].

3.2. **Some properties of differential operators.** We shall describe the actions of derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\pi\text{-Sen}}(V)$, we have

$$\nabla^{(0)}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(\pi)}(x) = \lim_{\beta \to 1} \frac{\beta(x) - x}{c(\beta)}.$$

By using these presentations, we compute the bracket $[\ ,\ ]$ of derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$.

Proposition 3.6. *On the differential module $D_{\pi\text{-Sen}}(V)$, we have $[\nabla^{(0)}, \nabla^{(\pi)}] = \nabla^{(\pi)}$.*

Proof. First, note that we have the relation $\gamma\beta = \beta^{\chi(\gamma)}\gamma$. Then, since we have

$$\lim_{h\to 0}\frac{a^{h+1}-a}{(h+1)-1} = a\log(a),$$

we obtain

$$\begin{aligned}
[\nabla^{(0)},\nabla^{(\pi)}](*) =& \lim_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}\lim_{\beta\to 1}\frac{\beta-1}{c(\beta)}(*) - \lim_{\beta\to 1}\frac{\beta-1}{c(\beta)}\lim_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}(*)\\
=& \lim_{\beta\to 1}\lim_{\gamma\to 1}\frac{\gamma\beta-\gamma-\beta+1}{(\chi(\gamma)-1)c(\beta)}(*) - \lim_{\beta\to 1}\lim_{\gamma\to 1}\frac{\beta\gamma-\gamma-\beta+1}{(\chi(\gamma)-1)c(\beta)}(*)\\
=& \lim_{\beta\to 1}\lim_{\gamma\to 1}\frac{\beta^{\chi(\gamma)}\gamma-\beta\gamma}{(\chi(\gamma)-1)c(\beta)}(*)\\
=& \lim_{\beta\to 1}\frac{\beta\log(\beta)}{c(\beta)}(*)\\
=& \nabla^{(\pi)}(*).
\end{aligned}$$

□

Proposition 3.7. *The action of the K_∞^{BK}-linear derivation $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is nilpotent.*

Proof. From the equality $\nabla^{(0)}\nabla^{(\pi)} - \nabla^{(\pi)}\nabla^{(0)} = \nabla^{(\pi)}$, we get $\nabla^{(0)}(\nabla^{(\pi)})^r - (\nabla^{(\pi)})^r\nabla^{(0)} = r(\nabla^{(\pi)})^r$ and $\mathrm{tr}(r(\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. Since the characteristic of K_∞^{BK} is 0, we obtain $\mathrm{tr}((\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the K_∞^{BK}-linear derivation $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is nilpotent. □

Proposition 3.8. *For an element $x \in D_{\pi\text{-Sen}}(V)$ such that $\nabla^{(0)}(x) = nx$ ($n \in \mathbb{Z}$), we have $\nabla^{(0)}(\nabla^{(\pi)}(x)) = (n+1)\nabla^{(\pi)}(x)$, that is, the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weight by 1.*

Proof. This follows easily from the relation $[\nabla^{(0)},\nabla^{(\pi)}] = \nabla^{(\pi)}$. □

There are many choices of a K_∞^{BK}-subspace of dimension $\dim_{\mathbb{Q}_p}V$ in $(\mathbb{C}_p\otimes_{\mathbb{Q}_p}V)^H$ equipped with derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$. The aim of this article is, however, to construct a differential module in $\mathbb{C}_p\otimes_{\mathbb{Q}_p}V$ which is closely related to the module $D_{\mathrm{st}}(V)$. Thus, the following proposition says that the choice $D_{\pi\text{-Sen}}(V)$ may be a reasonable one.

Proposition 3.9. *For a crystalline representation V of G_K, the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is trivial.*

Proof. In the case when V is a crystalline representation of G_K, we can take $\{\overline{f_j}\}_j$ as a basis of $D_{\pi\text{-Sen}}(V)$ over K_∞^{BK}. We can see that the action of Γ_{BK} on this basis is trivial and thus the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is trivial. □

Conversely, there is a semi-stable representation V of G_K such that the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is non-trivial. The next example is the prototype of such a semi-stable representation.

Example 3.10. Let V be a p-adic representation of G_K attached to the Tate curve $\overline{K}^*/\langle\pi\rangle$. We can take a basis $\{e, f\}$ of V over $\mathbb{Q}_p$ such that the action of $g \in G_K$ is given by

$$\begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $\{h_1 = 1 \otimes f, h_2 = 1 \otimes e\}$ ($\subset \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$) forms a basis of $D_{\pi\text{-Sen}}(V)$ over K_∞^{BK}. As indicated by Proposition 3.4, we have

$$\nabla^{(0)}(h_1) = 0 \quad \text{and} \quad \nabla^{(0)}(h_2) = h_2,$$

that is, the Hodge-Tate weights of V are $\{0, 1\}$. Furthermore, the action of $\nabla^{(\pi)}$ on this basis is given by

$$h_1 \xrightarrow{\nabla^{(\pi)}} h_2 \xrightarrow{\nabla^{(\pi)}} 0.$$

This means that the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$ is nilpotent (Proposition 3.7) and that the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weights by 1 (Proposition 3.8). Thus, we can know more than Hodge-Tate weights by using the K_∞^{BK}-vector space $D_{\pi\text{-Sen}}(V)$ equipped with $\nabla^{(\pi)}$.

4. Geometric aspect of $D_{\pi\text{-Sen}}(V)$

Let X be a proper smooth scheme over K. Then, it is known that the p-adic étale cohomology $V^m = H^m_{\text{ét}}(X \otimes_K \overline{K}, \mathbb{Q}_p)$ is a de Rham representation of G_K. Furthermore, due to the result of Berger, we can see that V^m is a potentially semi-stable representation of G_K. Let L/K be a finite field extension of K in $\overline{K}$ such that V^m is a semi-stable representation of G_L and let V_L^m denote the restriction of V^m to G_L.

In this section, we shall study the geometric aspect of $D_{\pi\text{-Sen}}(V_L^m)$ and see that the action of $\nabla^{(\pi)}$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem. First, by Proposition 3.4, we obtain the Γ_K-equivariant isomorphism of L_∞^{BK}-vector spaces $D_{\pi\text{-Sen}}(V_L^m) \simeq \bigoplus_{j=1}^{\dim_{\mathbb{Q}_p} V} L_\infty^{\mathrm{BK}}(n_j)$. With this presentation, define the subspace $D^{s,t}_{\pi\text{-Sen}}(V_L^m)$ of $D_{\pi\text{-Sen}}(V_L^m)$ to be $D^{m-t,t}_{\pi\text{-Sen}}(V_L^m) = \{x \in D_{\pi\text{-Sen}}(V_L^m) \,|\, \nabla^{(0)}(x) = tx\}$ ($t \in \mathbb{Z}$). It follows easily that we obtain the decomposition

$$D_{\pi\text{-Sen}}(V_L^m) = \bigoplus_{s+t=m} D^{s,t}_{\pi\text{-Sen}}(V_L^m).$$

The next proposition claims that the action of $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V_L^m)$ satisfies a formula similar to Griffiths transversality.

Proposition 4.1. *(Transversality) With notations as above, we have*

$$\nabla^{(\pi)}(D^{s,t}_{\pi\text{-Sen}}(V^m_L)) \subset D^{s-1,t+1}_{\pi\text{-Sen}}(V^m_L).$$

Proof. This follows easily from Proposition 3.8. □

By the same argument, we can see that an analogy of the local monodromy theorem holds for the L^{BK}_∞-vector space $D_{\pi\text{-Sen}}(V^m_L)$ equipped with $\nabla^{(\pi)}$.

Proposition 4.2. *(Local monodromy theorem) With notations as above, the L^{BK}_∞-linear operator $\nabla^{(\pi)}$ satisfies*

$$(\nabla^{(\pi)})^{m+1} \mid D_{\pi\text{-Sen}}(V^m_L) = 0.$$

Furthermore, if we put $h^{s,t} = \dim_{L^{\mathrm{BK}}_\infty} D^{s,t}_{\pi\text{-Sen}}(V^m_L)$ and define $h_m = \sup\{b-a \mid \forall i \in [a,b], h^{i,m-i} \neq 0\}$, we have

$$(\nabla^{(\pi)})^{h_m+1} \mid D_{\pi\text{-Sen}}(V^m_L) = 0.$$

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