# GENERALIZATION OF THE THEORY OF MIXED HODGE STRUCTURES AND ITS APPLICATION

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ABSTRACT. In this paper, we shall generalize the theory of mixed Hodge structures due to Deligne and obtain a subcategory GMHS in the category of mixed Hodge structures such that we have $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}, -) \neq 0$ in general.

## 1. INTRODUCTION

For a smooth projective scheme $X$ over $\mathbb{C}$, there exists a conjectural filtration $\mathrm{F}_M$ (called BBM filtration) on the Chow group $\mathrm{CH}^r(X, \mathbb{Q}) = \mathrm{CH}^r(X) \otimes \mathbb{Q}$ such that we have $\mathrm{Gr}^m_{\mathrm{F}_M}\mathrm{CH}^r(X, \mathbb{Q}) = \mathrm{Ext}^m_M(\mathbb{Q}, H^{2r-m}(X)(r))$. Here, $M$ is the conjectural category of mixed motives over $\mathrm{Spec}\,(\mathbb{C})$. On the other hand, by the realization functor $M \to \mathrm{MHS}$ from the category of mixed motives to that of mixed Hodge structures, we should have

$$\mathrm{Gr}^m_{\mathrm{F}_M}\mathrm{CH}^r(X, \mathbb{Q}) \to \mathrm{Ext}^m_{\mathrm{MHS}}(\mathbb{Q}, H^{2r-m}(X(\mathbb{C}), \mathbb{Q}(r))).$$

It is well-known, however, that the higher extension group $\mathrm{Ext}^m_{\mathrm{MHS}}(\mathbb{Q}, -)$ for $2 \leq m$ always vanishes and one cannot obtain any information about the graded piece $\mathrm{Gr}^m_{\mathrm{F}_M}\mathrm{CH}^r(X, \mathbb{Q})$ for $2 \leq m$ by using the extension of mixed Hodge structures.

In this paper, we shall generalize the theory of mixed Hodge structures due to Deligne and obtain a subcategory GMHS in the category of mixed Hodge structures such that we have $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}, -) \neq 0$ in general. Note that M.Asakura constructs another category (called the category of arithmetic Hodge structures) and shows that the higher extension group does not vanish in this category [A]. One will see that the category GMHS is an abelian category and that there is a forgetful functor $\mathcal{F}$ to the category MHS.

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## 2. Mixed Hodge structures

2.1. **Review of the classical theory.** For a compact Kähler manifold $X$, Hodge shows that there exists a decomposition

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is a complex subspace and satisfies the Hodge symmetry $H^{p,q}(X) = \overline{H^{q,p}(X)}$ ($-$ denotes the complex conjugation). This leads to the following definition.

**Definition 2.1.** An integral Hodge structure of pure weight $k$ is a free abelian group $H_{\mathbb{Z}}$ of finite type equipped with a decomposition

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} = \oplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is a complex subspace and satisfies the symmetry $H^{p,q} = \overline{H^{q,p}}$.

Let $H_{\mathbb{Z}}$ be an integral Hodge structure of pure weight $k$ and define a decreasing filtration (called Hodge Filtration) $F^{\cdot} H_{\mathbb{C}}$ by $F^p H_{\mathbb{C}} = \oplus_{p \leq r} H^{r,k-r}$. This filtration satisfies $H_{\mathbb{C}} = \mathrm{F}^p H_{\mathbb{C}} \oplus \overline{F^{k-p+1} H_{\mathbb{C}}}$ and determines the Hodge decomposition by the formula $H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}$. Let HS be the category of Hodge structures of pure weights: its object is given by Hodge structure $(H_{\mathbb{Z}}, F^{\cdot} H_{\mathbb{C}})$ of pure weight and its morphism is given by a morphism $f : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ which is compatible with the Hodge filtration $F^{\cdot}$.

For a general separated scheme $X$ of finite type over $\mathbb{C}$, the cohomology group $H^k(X, \mathbb{Z})$ does not carry the Hodge structure of pure weight in general. Then, Deligne shows that there exists an increasing filtration $W.$ (called weight filtration) on $H^k(X, \mathbb{Z})$ such that the Hodge filtration induced on $\mathrm{Gr}_r^W H^k(X, \mathbb{C})$ defines an integral Hodge structure of pure weight $k + r$ on $\mathrm{Gr}_r^W H^k(X, \mathbb{Z})$ ([D1], [D2]). This result leads to the following definition.

**Definition 2.2.** A mixed Hodge structure of weight $k$ is a free abelian group $H_{\mathbb{Z}}$ of finite type equipped with an increasing filtration (called weight filtration) $W.$ on $H_{\mathbb{Z}}$ and a decreasing filtration (called Hodge filtration) $F^{\cdot}$ on $H_{\mathbb{C}}$ such that the filtration induced by $F^{\cdot}$ on $\mathrm{Gr}_r^W H_{\mathbb{C}}$ defines a Hodge structure of pure weight $k + r$ on $\mathrm{Gr}_r^W H_{\mathbb{Z}}$.

Let MHS be the category of mixed Hodge structures: its object is given by a mixed Hodge structure $(H_{\mathbb{Z}}, F^{\cdot} H_{\mathbb{C}}, W.H_{\mathbb{Z}})$ and its morphism is given by a morphism $f : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ which is compatible with filtrations $F^{\cdot}$ and $W.$. It is known that the category MHS is an abelian category ([D1], (2.3.5)).

2.2. **Generalized mixed Hodge structures.** In this subsection, we shall generalize the theory of mixed Hodge structures due to Deligne. Let $U$ be a smooth and separated scheme of finite type over $\mathbb{C}$ and $X$ be a smooth compactification of $U$. By using subschemes on $U$ and $D = X \backslash U$, we shall introduce two structures on cohomology groups: $z$-structures and $w$-structures. These lead to the generalization of the theory of mixed Hodge structures.

2.2.1. *$z$-structures on cohomology groups.* For a subscheme $V$ on $U$, let $\vec{r} = (r_1, \cdots, r_l)$ denote a basis of $\mathrm{Im}\,(H^n_V(U,\mathbb{Q}) \to H^n(U,\mathbb{Q}))$ over $\mathbb{Q}$. Choose elements $\vec{s} = (s_1, \cdots, s_m)$ of $H^n(U,\mathbb{Q})$ such that $\{\vec{r}, \vec{s}\}$ forms a basis of $H^n(U,\mathbb{Q})$ over $\mathbb{Q}$. Then, define an involution $z_{V,\vec{r},\vec{s}}$ on $H^n(U,\mathbb{Q})$ by the formula

$$z_{V,\vec{r},\vec{s}}(\sum_{i=1}^{l} a_i r_i + \sum_{j=1}^{m} b_j s_j) = -\sum_{i=1}^{l} a_i r_i + \sum_{j=1}^{m} b_j s_j \quad (a_i, b_j \in \mathbb{Q}).$$

2.2.2. *$w$-structures on cohomology groups.* Since we assume that $U$ is a smooth and separated scheme of finite type over $\mathbb{C}$, it is a Zariski open set in a complete scheme $X$ ([N]). Furthermore, we assume that $X$ is smooth projective and that the complement $D = X \backslash U$ is a globally normal crossing divisor, that is, we have $D = \cup_{i\in I} D_i$ where each $D_i \subset X$ is a smooth hypersurface and the intersection of hypersurfaces is transverse ([H]).

*Notation .* For a subset $K \subset I$, put $D_K = \cap_{i\in K} D_i$ and let $D^{(k)}$ denote the disjoint union of $D_K$ where $K$ runs through subsets of $I$ of cardinal $k$. Set $D^{(0)} = X$.

For the weight spectral sequence ${}_W E$ associated to the weight filtration $W.$, we have ${}_W E_1^{p,q} \simeq H^{2p+q}(D^{(-p)},\mathbb{C})$ and its differential $d_1$ is given by

$$\begin{array}{ccc} H^{2p+q}(D^{(-p)},\mathbb{C}) & \xrightarrow{d_1} & H^{2p+q+2}(D^{(-p-1)},\mathbb{C}) \\ \Big\| & & \Big\| \\ \bigoplus_{|K|=-p} H^{2p+q}(D_K,\mathbb{C}) & \xrightarrow{d_1} & \bigoplus_{|L|=-p-1} H^{2p+q+2}(D_L,\mathbb{C}) \end{array} \tag{2.1}$$

where $d_1$ has the component $d_{1K}^{L}$ equal to zero for $L \not\subset K$ and equal to $(-1)^{q+s} j^L_{K*}$ for $K = \{i_1 < \cdots < i_p\}$ and $L = K \backslash \{i_s\}$ where $j^L_{K*}$ denotes the Gysin map corresponding to the inclusion $j^L_K : D_K \hookrightarrow D_L$. Due to the result of Deligne, this spectral sequence degenerates at $E_2$ and we obtain ${}_W E_2^{p,q} = \mathrm{Gr}^W_{-p} H^k(U,\mathbb{C})$. For each subscheme $V'$ on $D$, we shall define a $\mathbb{C}$-linear involution $w_{V'}$ on $\mathrm{Gr}^W_{-p} H^k(U,\mathbb{C})$. If we have $V' \not\subset D^{(-p)}$, put $w_{V'}(c) = c$. Now, assume that $V'$ is a subscheme on $D^{(-p)}$ and then there is a natural morphism

$$\psi_{V'} : H^{2p+q}_{V'}(D^{(-p)},\mathbb{C}) \to H^{2p+q}(D^{(-p)},\mathbb{C}) \simeq {}_W E_1^{p,q} \twoheadrightarrow {}_W E_1^{p,q}/\mathrm{Im}\,({}_W E_1^{p-1,q}).$$

Let $\vec{r} = (r_1, \cdots, r_l)$ denote a basis of $\mathrm{Im}\,(\psi_{V'}) \cap {}_W E_2^{p,q}$ over $\mathbb{C}$. Choose elements $\vec{s} = (s_1, \cdots, s_m)$ of ${}_W E_2^{p,q}$ such that $\{\vec{r}, \vec{s}\}$ forms a basis of ${}_W E_2^{p,q}$ over $\mathbb{C}$. Then, define an involution $w_{V',\vec{r},\vec{s}}$ on $\mathrm{Gr}^W_{-p} H^k(U,\mathbb{C})$ by the formula

$$w_{V',\vec{r},\vec{s}}(\sum_{i=1}^{l} a_i r_i + \sum_{j=1}^{m} b_j s_j) = -\sum_{i=1}^{l} a_i r_i + \sum_{j=1}^{m} b_j s_j \quad (a_i, b_j \in \mathbb{C}).$$

2.2.3. *Category of generalized mixed Hodge structures.* The results of preceding subsections lead to the following definition.

**Definition 2.3.** Let $U$ be a smooth and separated scheme of finite type over $\mathbb{C}$ and $X$ be a smooth compactification of $U$ such that $D = X\backslash U$ is a globally normal crossing divisor. A generalized mixed Hodge structure consists of $(H_{\mathbb{Z}}, F^{\cdot}, W_{\cdot}, \{z_V\}_{V\subset U}, \{w_{V'}\}_{V'\subset D})$ where

- the triple $(H_{\mathbb{Z}}, F^{\cdot}, W_{\cdot})$ is a mixed Hodge structure,
- $z_V$ denotes a $\mathbb{C}$-linear involution on $H_{\mathbb{C}}$ for each subscheme $V$ on $U$,
- $w_{V'}$ denotes a $\mathbb{C}$-linear isomorphism of $H_{\mathbb{C}}$ such that the induced action on $\mathrm{Gr}^W_m H_{\mathbb{C}}$ is an involution for each subscheme $V'$ on $D$.

Let GMHS denote the category of generalized mixed Hodge structures: its object is given by a generalized mixed Hodge structure and its morphism between $\big\{(H^i_{\mathbb{Z}}, F^{\cdot}, W_{\cdot}, \{z_{V_i}\}_{V_i\subset U_i}, \{w_{V'_i}\}_{V'_i\subset D_i})\big\}_{i=1,2}$ is given by the pair of a morphism of mixed Hodge structures $f : H^1_{\mathbb{Z}} \to H^2_{\mathbb{Z}}$ and a morphism of schemes $g : X_2 \to X_1$ where $X_i$ denotes a smooth compactification of $U_i$ such that we have $D_i = X_i\backslash U_i$. Furthermore, assume that this pair of morphisms satisfies the compatible condition $f \circ x = y \circ f$ where

$$\begin{cases} x = z_{V_1}, & y = z_{V_2} \quad \text{if } g(U_2 - V_2) \subset U_1 - V_1 \text{ and } g : V_2 \simeq V_1,\\ x = z_{V_1}, & y = w_{V'_2} \quad \text{if } g(D_2 - V'_2) \subset U_1 - V_1 \text{ and } g : V'_2 \simeq V_1,\\ x = w_{V'_1}, & y = z_{V_2} \quad \text{if } g(U_2 - V_2) \subset D_1 - V'_1 \text{ and } g : V_2 \simeq V'_1,\\ x = w_{V'_1}, & y = w_{V'_2} \quad \text{if } g(D_2 - V'_2) \subset D_1 - V'_1 \text{ and } g : V'_2 \simeq V'_1. \end{cases}$$

One can verify that the category GMHS is an abelian category and that there is a forgetful functor $\mathcal{F}$ to the category MHS.

## 3. Extension groups $\mathrm{Ext}^m$

For a smooth projective scheme $X$ over $\mathbb{C}$, the conjectural filtration $\mathrm{F}_M$ on the Chow group $\mathrm{CH}^r(X, \mathbb{Q})$ should satisfy $\mathrm{Gr}^m_{\mathrm{F}_M}\mathrm{CH}^r(X, \mathbb{Q}) = \mathrm{Ext}^m_M(\mathbb{Q}, H^{2r-m}(X)(r))$. Here, $M$ is the conjectural category of mixed motives over Spec ($\mathbb{C}$). On the other hand, by the realization functor $M \to$ MHS, we should have

$$\mathrm{Gr}^m_{\mathrm{F}_M}\mathrm{CH}^r(X, \mathbb{Q}) \to \mathrm{Ext}^m_{\mathrm{MHS}}(\mathbb{Q}, H^{2r-m}(X(\mathbb{C}), \mathbb{Q}(r))).$$

From the right exactness of $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, -)$, however, it follows that the higher extension group $\mathrm{Ext}^m_{\mathrm{MHS}}(\mathbb{Q}, -)$ for $2 \le m$ always vanishes [C]. In this section, we shall introduce the one dimensional vector space $\mathbb{Q}_M$ over $\mathbb{Q}$ equipped with generalized mixed Hodge structures and shall construct an example which shows that $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M, -)$ does not vanish in general.

3.1. **Definition of $\mathbb{Q}_M$.** Let $U$ be a smooth and separated scheme of finite type over $\mathbb{C}$ and $X$ be a smooth compactification of $U$ such that $D = X\backslash U$ is a globally normal crossing divisor. Define $S(U,D)$ to be the set of subschemes $(\{V\}_{V\subset U}, \{V'\}_{V'\subset D})$ where $V$ (resp. $V'$) runs through any subscheme on $U$ (resp. $D$).

**Definition 3.1.** With notations as above, for a subset $M(U,D)$ of $S(U,D)$, let $\mathbb{Q}_{M(U,D)}$ denote the one dimensional vector space over $\mathbb{Q}$ equipped with the generalized mixed Hodge structure $(\mathbb{Q}, F^{\cdot}, W_{\cdot}, \{z_V\}_{V\subset U}, \{w_{V'}\}_{V'\subset D})$ where the mixed Hodge structure is trivial and involutions $(\{z_V\}_{V\subset U}, \{w_{V'}\}_{V'\subset D})$ act on $\mathbb{Q}_{M(U,D)}$ by

$$\begin{cases} z_V(a) = -a \text{ if } V \in M(U,D), & z_V(a) = a \text{ if } V \notin M(U,D) \\ w_{V'}(a) = -a \text{ if } V' \in M(U,D), & w_{V'}(a) = a \text{ if } V' \notin M(U,D) \end{cases}$$

**Example 3.2.** Let $X$ be a smooth projective scheme over $\mathbb{C}$ and let $cl : \mathrm{CH}^j(X,\mathbb{Q}) \to H^{2j}(X(\mathbb{C}),\mathbb{C})$ denote the cycle map. Then, the classical Hodge conjecture states that this cycle map has the image

$$H^{j,j}(X) \cap H^{2j}(X(\mathbb{C}),\mathbb{Q}) = \mathrm{Ext}^0_{\mathrm{HS}}(\mathbb{Q}, H^{2j}(X(\mathbb{C}),\mathbb{Q})).$$

Assume that the classical Hodge conjecture holds. Then, we can write $\mathrm{Ext}^0_{\mathrm{HS}}(\mathbb{Q}, H^{2j}(X(\mathbb{C}),\mathbb{Q}))$ in terms of generalized mixed Hodge structures

$$\mathrm{Ext}^0_{\mathrm{HS}}(\mathbb{Q}, H^{2j}(X(\mathbb{C}),\mathbb{Q})) = \bigoplus_{M\subset S(X,\phi)} \mathrm{Ext}^0_{\mathrm{GMHS}}(\mathbb{Q}_M, H^{2j}(X(\mathbb{C}),\mathbb{Q})).$$

*Proof.* It suffices to show that we have LHS $\subset$ RHS. Note that the cohomology group $H^{2j}(X(\mathbb{C}),\mathbb{Q})$ is equipped with the involution $z_V$ for each subscheme $V$ on $X$ through $H^{2j}_V(X,\mathbb{Q}) \to H^{2j}(X,\mathbb{Q})$. Take an element $f_a$ $(: 1 \mapsto a)$ of $\mathrm{Ext}^0_{\mathrm{HS}}(\mathbb{Q}, H^{2j}(X(\mathbb{C}),\mathbb{Q}))$. By the assumption, there is an element $\widetilde{a}$ of $\mathrm{CH}(X,\mathbb{Q})$ such that we have $f_a = cl(\widetilde{a})$. We can write this cycle $\widetilde{a}$ as $\sum_{k=1}^m n_k \widetilde{a}_k$ $(n_k \in \mathbb{Z})$ where $\{\widetilde{a}_k\}_{k=1}^m$ denote subschemes on $X$. Then, the element $f_a = \sum_{k=1}^m n_k cl(\widetilde{a}_k)$ is contained in $\bigoplus_{k=1}^m (\bigoplus_{M_k} \mathrm{Ext}^0_{\mathrm{GMHS}}(\mathbb{Q}_{M_k}, H^{2j}(X(\mathbb{C}),\mathbb{Q})))$ where $M_k$ runs through any set containing $\big(\{\widetilde{a}_k\}, \{\phi\}\big)$ and thus we obtain LHS $\subset$ RHS. $\square$

**Example 3.3.** With notations as in the previous example, define $\mathrm{CH}^j(X,\mathbb{Q})_{\mathrm{hom}} = \{\alpha \in \mathrm{CH}^j(X,\mathbb{Q}) \,|\, cl(\alpha) = 0\}$. Then, we have the Abel-Jacobi map

$$cl' : \mathrm{CH}^j(X,\mathbb{Q})_{\mathrm{hom}} \to \frac{H^{2j-1}(X(\mathbb{C}),\mathbb{C})}{F^j H^{2j-1}(X(\mathbb{C}),\mathbb{C}) \oplus H^{2j-1}(X(\mathbb{C}),\mathbb{Q})}.$$

One can see that the target of this map is isomorphic to $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^{2j-1}(X,\mathbb{Q}))$. First, we shall review the construction of the extension class given by the Abel-Jacobi map. Let $v$ denote an element of $\mathrm{CH}^j(X,\mathbb{Q})_{\mathrm{hom}}$ and $V$ be the support of

$v$. Put $U = X\backslash V$. Then, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2j-1}(X,\mathbb{Q}) & \longrightarrow & H^{2j-1}(U,\mathbb{Q}) & \longrightarrow & H^{2j}_V(X,\mathbb{Q}) & \longrightarrow & H^{2j}(X,\mathbb{Q}) \\ & & \| & & \cup\uparrow & & h_v\uparrow & & \\ 0 & \longrightarrow & H^{2j-1}(X,\mathbb{Q}) & \longrightarrow & E & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \end{array}$$

where $h_v$ denotes the map $\mathbb{Q} \to H^{2j}_V(X,\mathbb{Q}) : 1 \mapsto v$ and the bottom exact sequence is obtained by pull-back via $h_v$. We can verify that the extension class $E$ of this exact sequence is the image of $v$ under the Abel-Jacobi map ([J1], 9.4). Now, let us see the bottom exact sequence of the diagram above in terms of generalized mixed Hodge structures. We can write the cycle $v$ as $\sum_{k=1}^m n_k v_k$ ($n_k \in \mathbb{Z}$) where $\{v_k\}_{k=1}^m$ denote subschemes on $X$. We will denote $\mathbb{Q}$ in the diagram above by $\mathbb{Q}_v$ and fix a basis $1_v$ of $\mathbb{Q}_v$ over $\mathbb{Q}$. Since the involution $v \mapsto -v$ on $H^{2j}_V(X,\mathbb{Q})$ should correspond to the involution $f_v : 1_v \mapsto -1_v$, it is natural to think that $\mathbb{Q}_v$ is contained in $\mathbb{Q}_N = \bigoplus_{k=1}^m(\bigoplus_{M_k}\mathbb{Q}_{M_k})$ where $M_k$ runs through any set of $S(X,\phi)$ containing $(\{v_k\},\{\phi\})$. Since the extension class $E$ is the image of $v$ under the Abel-Jacobi map, it is compatible with the action of $f_v$ induced by $z$-structures on $\mathbb{Q}_N$. Furthermore, $E$ is clearly compatible with $z$-structures on $H^{2j-1}(X,\mathbb{Q})$. Thus, we can regard the extension class $E$ as an element of $\bigoplus_{M\subset S(X,\phi)} \mathrm{Ext}^1_{\mathrm{GMHS}}(\mathbb{Q}_M, H^{2j-1}(X,\mathbb{Q}))$.

3.2. **Non-vanishing of $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M,-)$.** In this section, we shall see that the higher extension group $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M,-)$ does not vanish in general. First, we shall recall the Yoneda extension class. Let $A$ denote an abelian category. For objects $M$ and $N$ of $A$, an element $C(E)$ of $\mathrm{Ext}^n_A(M,N)$ (called the Yoneda extension class) is given by an exact sequence

$$E : 0 \to N \to R_n \to R_{n-1} \to \cdots \to R_1 \to M \to 0.$$

Let $E' : 0 \to N \to R'_n \to R'_{n-1} \to \cdots \to R'_1 \to M \to 0$ be another extension. Then, we have $C(E) = C(E')$ if and only if there exists an extension $E'' : 0 \to N \to R''_n \to R''_{n-1} \to \cdots \to R''_1 \to M \to 0$ such that we have the following commutative diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & R_n & \longrightarrow & R_{n-1} & \longrightarrow & \cdots & \longrightarrow & R_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & & & \uparrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & R''_n & \longrightarrow & R''_{n-1} & \longrightarrow & \cdots & \longrightarrow & R''_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & R'_n & \longrightarrow & R'_{n-1} & \longrightarrow & \cdots & \longrightarrow & R'_1 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

**Example 3.4.** We shall construct an example which shows that $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M,-)$ does not vanish. Let us consider the following exact sequence in the category of generalized mixed Hodge structures

$$E : 0 \to S \xrightarrow{i} T \xrightarrow{j} V \xrightarrow{k} \mathbb{Q}_M \to 0.$$

Here,

- $S$ is a 2-dimensional vector space over $\mathbb{Q}$ equipped with the Hodge structure of pure weight $-1$. For a smooth projective curve $X$ over $\mathbb{C}$, assume that $S$ is endowed with the trivial $z$-structure on $X$ and the trivial $w$-structure on $\phi$ $(= X \backslash X)$.

- $T$ is a 3-dimensional vector space over $\mathbb{Q}$ equipped with the mixed Hodge structure such that $\mathrm{Gr}_0^W(T)$ has the Hodge structure of pure weight $-1$ and $\mathrm{Gr}_1^W(T)$ has the Hodge structure of pure weight 0. For two points $\{D_i\}_{i=1,2}$ on $X$, assume that $T$ is equipped with the trivial $z$-structure on $U = X\backslash\{D_i\}_{i=1,2}$ and with the $w$-structure on $\{D_i\}_{i=1,2}$. Here, the action of $\{w_{D_i}\}_{i=1,2}$ on a basis $\{e, s_1, s_2\}$ of $T$ over $\mathbb{Q}$ is given by

$$w_{D_i}\begin{pmatrix} e \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e \\ s_1 \\ s_2 \end{pmatrix}$$

  where $\{s_1, s_2\}$ denotes a basis of $i(S)$ over $\mathbb{Q}$. Note that these actions induce (trivial) involutions on $\mathrm{Gr}_i^W(T)$ $(i = 0, 1)$.

- $V$ is a 2-dimensional vector space over $\mathbb{Q}$ equipped with the Hodge structure of pure weight 0. Assume that $V$ is equipped with the trivial $w$-structure on $\phi$ and with the $z$-structure on $\{D_i\}_{i=1,2}$ such that the action of $\{z_{D_i}\}_{i=1,2}$ on a basis $\{\alpha, \beta\}$ of $V$ over $\mathbb{Q}$ is given by

$$z_{D_i}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c(D_i) & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

  where $\{c(D_i)\}_{i=1,2}$ satisfy $c(D_1) \neq c(D_2)$. For a subscheme $D'$ on $X$ other than $\{D_i\}_{i=1,2}$, assume that the action of $z_{D'}$ on $V$ is trivial. Then, we can consider that $V$ is also equipped with the $z$-structure on $X$ and the trivial $w$-structure on $\phi$.

- $M = (\{D_i\}_{i=1,2}, \{\phi\}) \subset S(X, \phi)$, that is, $\mathbb{Q}_M$ is endowed with the non-trivial action of $z_{D_i}$ and the trivial $w$-structure on $\phi$.

On the other hand, one can verify that the exact sequence $E' : 0 \to S \to S \to \mathbb{Q}_M \to \mathbb{Q}_M \to 0$ in GMHS gives a trivial Yoneda extension class. Thus, it suffices to show that we have $C(E) \neq C(E')$ in $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M, S)$. Assume that there exists an exact sequence $0 \to S \to T' \to V' \to \mathbb{Q}_M \to 0$ in GMHS such that we have the following commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & S & \xrightarrow{i} & T & \xrightarrow{j} & V & \xrightarrow{k} & \mathbb{Q}_M & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & S & \longrightarrow & T' & \xrightarrow{h} & V' & \xrightarrow{h'} & \mathbb{Q}_M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \| & & \\
0 & \longrightarrow & S & \longrightarrow & S & \longrightarrow & \mathbb{Q}_M & \longrightarrow & \mathbb{Q}_M & \longrightarrow & 0.
\end{array} \tag{3.1}
$$

First, note that, since the morphism $T' \xrightarrow{f'} S$ has a section, it induces the splitting $T' = S \oplus T'' = S \oplus \mathrm{Ker}\,(f')$ of generalized mixed Hodge structures. Now, we shall fix some notations. Let $\{c_j\}_{j=1}^n$ denote a basis of $T''$ over $\mathbb{Q}$ and put

$$
f_{|T''} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} p_1 & t_1 & u_1 \\ \vdots & \vdots & \vdots \\ p_n & t_n & u_n \end{pmatrix} \begin{pmatrix} e \\ s_1 \\ s_2 \end{pmatrix} \tag{3.2}
$$

where $\{p_j, t_j, u_j\}_{j=1}^n$ denotes elements of $\mathbb{Q}$. Then, it follows that $h(T'')$ is a $n$-dimensional subvector space of $V'$ spanned by the image $\{d_j = h(c_j)\}_{j=1}^n$ over $\mathbb{Q}$. Thus, if we choose an element $v$ of $V' \backslash h(T'')$, the elements $\{v, d_j\}_{j=1}^n$ form a basis of $V'$ over $\mathbb{Q}$. Define

$$
g(v) = p\alpha + q\beta \quad \text{and} \quad g(d_j) = p_j'\alpha + q_j'\beta.
$$

Then, since we have $k \circ g = h'$ by the commutative diagram, we obtain $q \neq 0$. Furthermore, since we also have $j \circ f = g \circ h$ by the commutative diagram and the action of $\{z_{D_i}\}_{i=1,2}$ on the image of $j$ in $V$ is given by 1, it follows that we have $p_j' = p_j$ and $q_j' = 0$ for all $1 \leq j \leq n$. Thus, we can write

$$
g \begin{pmatrix} v \\ d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} p & q \\ p_1 & 0 \\ \vdots & \vdots \\ p_n & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (q \neq 0). \tag{3.3}
$$

We shall show that we have $p_j = 0$ $(1 \leq j \leq n)$. For simplicity, assume that the weight filtration $W_jT''$ (resp. $W_{j+1}T''$) of $T''$ is spanned by $\{c_{l+1}, \cdots, c_n\}$ (resp. $\{c_k, \cdots, c_n\}$) and that the quotient $\mathrm{Gr}_{j+1}^W T''$ has the Hodge structure of pure weight 0. Then, by the argument of weights, it follows that we have

$$
p_j = 0 \quad (1 \leq j \leq k-1 \text{ and } l+1 \leq j \leq n).
$$

In particular, it follows from (3.2) that the image of $W_jT''$ under $f$ is contained in $i(S)$. By the commutative diagram, this means that we have $f_{|W_jT''} = 0$. Thus, we obtain

$$
t_j = 0 \quad \text{and} \quad u_j = 0 \quad (l+1 \leq j \leq n).
$$

On the other hand, since $T''$ is an object of GMHS, there exist actions $\{x_{D_i}\}_{i=1,2}$ on $T''$ which are compatible with the actions of $\{w_{D_i}\}_{i=1,2}$ on $T$, that is, these satisfy $w_{D_i}f = fx_{D_i}$. Note that these actions induce involutions on $\mathrm{Gr}_{j+1}^W T''$

by definition. Let $\vec{c}$ be the column vector ${}^t(c_k, \cdots, c_n)$ and put $x_{D_i}(\vec{c}) = R_i\vec{c}$. Furthermore, $R'_i$ denotes the submatrix of $R_i$ which represents the residual action of $x_{D_i}$ modulo $W_jT''$. Since we have $fx_{D_i}(\vec{c}) = w_{D_i}f(\vec{c})$ and $p_j = t_j = u_j = 0$ $(l+1 \leq j \leq n)$, it follows that we obtain

$$R'_i \begin{pmatrix} p_k e + t_k s_1 + u_k s_2 \\ \vdots \\ p_l e + t_l s_1 + u_l s_2 \end{pmatrix} = \begin{pmatrix} p_k e + (-p_k + t_k)s_1 + u_k s_2 \\ \vdots \\ p_l e + (-p_l + t_l)s_1 + u_l s_2 \end{pmatrix}.$$

If we put $\vec{p} ={}^t (p_k, \cdots, p_l)$ and $\vec{t} ={}^t (t_k, \cdots, t_l)$, this leads to $(R'_i - E)\vec{p} = 0$ and $(R'_i - E)\vec{t} = -\vec{p}$. Since $R'_i$ denotes the matrix of the involution $x_{D_i}$ on $\mathrm{Gr}^W_{j+1}T''$, we have $(R'_i + E)(R'_i - E) = 0$ and thus $(R'_i + E)\vec{p} = -(R'_i + E)(R'_i - E)\vec{t} = 0$. Therefore, it follows that $\vec{p}$ is the zero-vector and that we obtain

$$p_j = 0 \quad (1 \leq j \leq n).$$

Since $V'$ is also an object of GMHS, there exist actions $\{y_{D_i}\}_{i=1,2}$ on $V'$ which are compatible with the actions of $\{z_{D_i}\}_{i=1,2}$ on $V$, that is, these satisfy $z_{D_i}g = gy_{D_i}$. Put $y_{D_i}(v) = a_0 v + \sum_{j=1}^n a_j d_j$. Since we have the formula (3.3) and $p_j = 0$ for $1 \leq j \leq n$, it follows that we obtain $gy_{D_i}(v) = a_0(p\alpha + q\beta)$. On the other hand, since we have $z_{D_i}g(v) = z_{D_i}(p\alpha + q\beta) = p\alpha + q(c(D_i)\alpha - \beta)$, the compatibility leads to

$$a_0 = -1 \quad \text{and} \quad c(D_i) = -\frac{2p}{q} \quad (i = 1, 2).$$

Here, note that we have $q \neq 0$ by (3.3). This means that $c(D_i)$ does not depend on $D_i$ and that this contradicts the assumption $c(D_1) \neq c(D_2)$. Thus, the Yoneda extension class $C(E)$ given by $E$ is non-trivial in $\mathrm{Ext}^2_{\mathrm{GMHS}}(\mathbb{Q}_M, S)$.

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