ARITHMETIC PROPERTY OF THE DIFFERENTIAL CALCULUS

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1. Introduction

For example, when we drive from A to B, it takes an infinite number of measurements to work out the precise path. In this paper, however, if everything is written in polynomial form, we shall illustrate that these measurements can be done a finite number of times by using the discrete property of the derivative. This means that, for example, a finite number of Orbis can be used to enforce all speed violations. Apart from the analogy, if we apply this arithmetic property of derivative to number theory, this idea is nothing but defining the derivative in a discrete space over $\mathbb{Z}$.

2. Discrete property of the derivatives

One of the aims of the differential calculus is the linear approximation of differentiable objects. As a matter of course, it is impossible to determine the general curve by a finite number of tangent lines. For curves given by polynomials, the situation changes completely. The following theorem is the main tool in this paper.

Theorem 2.1. *Let $f(x), g(x)$ be the polynomials in x and assume that the curves $C_1 : y = f(x)$, $C_2 : y = g(x)$ on the xy-plain have, at least, one common point. Further, assume that the slopes of the tangent lines of C_1 and C_2 are the same at the sufficiently many (finite) $x = \alpha$ and then we have $f(x) = g(x)$.*

Proof. It can be obtained as a consequence of Van der Monde's determinant but we shall give a proof by using a rigidity of polynomials. Put $h(x) = f'(x) - g'(x)$ and assume that the degree of $h(x)$ is n. By the assumption, we have $h(\alpha) = 0$ for the sufficiently many $x = \alpha$ and since $h(x)$ is polynomial and the number of the solutions of $h(x) = 0$ is at most n, it follows that we have $h(x) \equiv 0$, that is, $f'(x) \equiv g'(x)$. Further, since the curves C_1 and C_2 have one common point, this means that we have $f(x) = g(x)$ without an integration constant. $\square$

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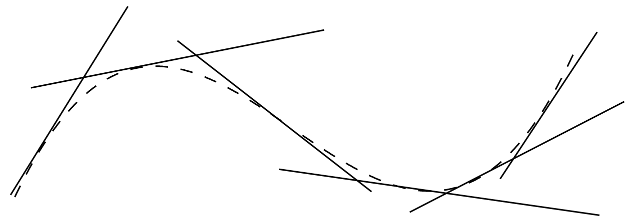


Determined by a finite number of the slopes of the tangent lines.

Note that the theorem asserts that the curve defined by a polynomial is determined not only by a finite number of the coordinates of tangent points but also by a finite number of the slopes of the tangent lines up to an integration constant.

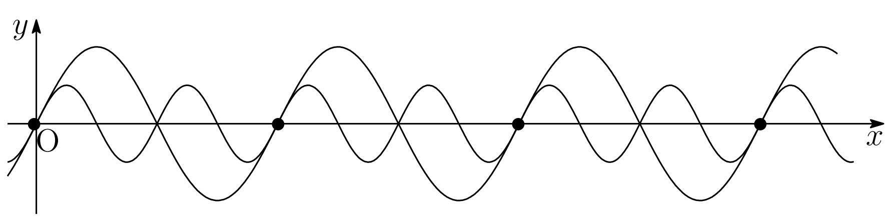
Example 2.2. We shall use a simple example to show how to explicitly determine the polynomial when we already know the degree of the polynomial. Let $f(x) = ax^3+bx^2+cx+d$ be the polynomial in x of degree 3 with $f'(0) = -8$, $f'(1) = -2$ and $f'(2) = 16$ and assume that $y = f(x)$ passes through the origin. Then, we have

$$c = -8, \quad 3a + 2b + c = -2, \quad 12a + 4b + c = 16$$

and it follows that we have $f(x) = 2x^3 - 8x$. As can be seen from this example, if the degree of the polynomial is known a priori, it is a simple linear equation problem.

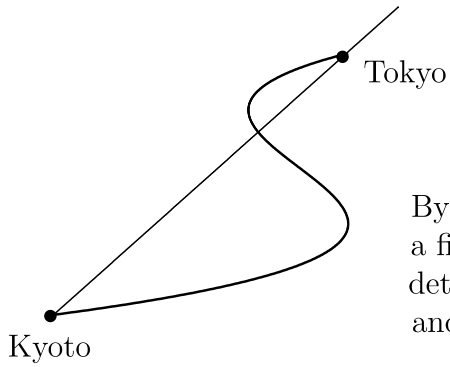
Example 2.3. On the other hand, without the assumption that it is polynomial, it is easy to make a counterexample without the use of topology. Take $f(x) = \sin x$ and $g(x) = \frac{1}{2}\sin 2x$ and then we have

$$f'(x) = \cos x,\ g'(x) = \cos 2x \implies f'(2m\pi) = g'(2m\pi) \quad (\forall m \in \mathbb{Z}).$$



Graphs of $y = \sin x$ and $y = \frac{1}{2}\sin 2x$ (the same tangent line at $\bullet$)

Example 2.4. Let $(x, y) = (x(t), y(t))$ be the position with respect to time t and assume that $x(t)$ and $y(t)$ are polynomials in t. In this case, the slope of the tangent line is nothing but the velocity. Suppose that, for example, we start from Kyoto and arrive at Tokyo and if enough velocity vectors $(x'(t), y'(t))$ are measured along the way, it can be uniquely determined which route has been followed.



By simply measuring the velocity vectors a finite number of times, it is possible to determine which route has been followed and at what speed.

It is not realistic, but if you drive a "polynomial car", it is useless to try to slow down in front of an Orbis since the entire path and its velocity can be calculated and thus a finite number of Orbis can be used to enforce all speed violations. Notice that we do not need data on where the measurements are taken, but only on when they are taken.

In connection with number theory, let us consider a discrete space composed entirely of lattice points. Since we can only perform a finite number of operations, we assume that a function f on the discrete space is a mapping of a finite number of x_i to a finite number of points y_i $(1 \leqq i \leqq n)$. Then, it is well-known in the elementary algebraic geometry that, by the Vandermonde's determinant, the polynomial $f(x)$ of degree $n-1$ or less is uniquely determined. Thus, a function on the discrete space can be represented by a polynomial. It has been one of goals in mathematics to investigate the "shape" of a discrete object by differentiating it, but as the above the theorem shows, it is determined from the values of the derivatives at the finitely many lattice points.